Power Series :

DEF. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

or

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

coefficients

"centered at a", or "about", or "in (x−a)"

note: ① $(x-a)^0 = 1$ even if $x=a$

② A power series may or may not converge for various values of x

ex/ $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$ (all $c_n = 1$)

This converges for $|x| < 1$ and diverges for $|x| \geq 1$ by GST

③ The sum of the series is a function of x

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

(like a polynomial with infinitely terms)

whose domain is all x-values where the series converges

ex/ ① For what x-values is the series $\sum_{n=0}^{\infty} n!\, x^n$ convergent?

ratio test

$= 1 + x + 2x^2 + 6x^3 + \cdots$

Ratio Test : $x=0$: $\boxed{\text{CONV}}$ and $f(0)=1$

$0!=1$ $1!=1$ $2!=2\cdot1=1$

$3!=3\cdot2\cdot1$

$x\neq 0$:

$a_n = n!\,x^n$

$a_{n+1} = (n+1)!\,x^{n+1}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}\left|(n+1)\,x\right| \underset{x\neq0}{=} \infty > 1 \qquad \boxed{\text{DIV}}$$

So the series converges for $x=0$ and no other x-values

② For which x-values does $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges?

Ratio test : $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

for all x

So the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x in the interval $(-\infty,\infty)$

③ For what x-values does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ Converge?

ratio test :

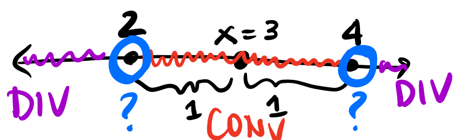
$$L = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(n+1)} \cdot \frac{n}{(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \cdot \frac{n}{n+1} \right|$$

a_{n+1} $\qquad$ a_n

$$\frac{(x-3)^{n+1}}{(x-3)^n} = \frac{\cancel{(x-3)^n}(x-3)^1}{\cancel{(x-3)^n}}$$

$$= \lim_{n\to\infty} \left| (x-3)\left(\frac{n}{n+1}\right) \right| = |x-3|$$

1



So $L<1$ if $|x-3|<1$

radius of conv.

$-1 < x-3 < 1$
$+3 \quad +3 \quad +3$
$2 < x < 4$
CONV

$L=1$ if $|x-3|=1$

$x-3=1 \rightarrow x=4$
$x-3=-1 \rightarrow x=2$
INCONCLUSIVE

$L>1$ if $|x-3|>1$

$x<2$ or $x>4$
DIVERGES

So the series converges when $2<x<4$ and diverges when $x<2$ or $x>4$ and we need to check the endpts. $x=2$, $x=4$

endpts : $x=2$: $\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ CONV by AST

$\underline{x=4}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ DIV harmonic

$1^n = 1$

So the series converges for x in the interval

$[2, 4)$ interval of convergence

include 2 exclude 4