Math 143 – part 2

by Amanda Tucker

Alternating series estimation theorem.

If $s = \sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n \geq 0$ is the sum of a convergent alternating series then

$$|R_n| = |s - s_n| \leq |s_{n+1} - s_n| = a_{n+1} \quad \text{so} \quad \boxed{|R_n| \leq a_{n+1}}$$

The n^{th} remainder "the error"

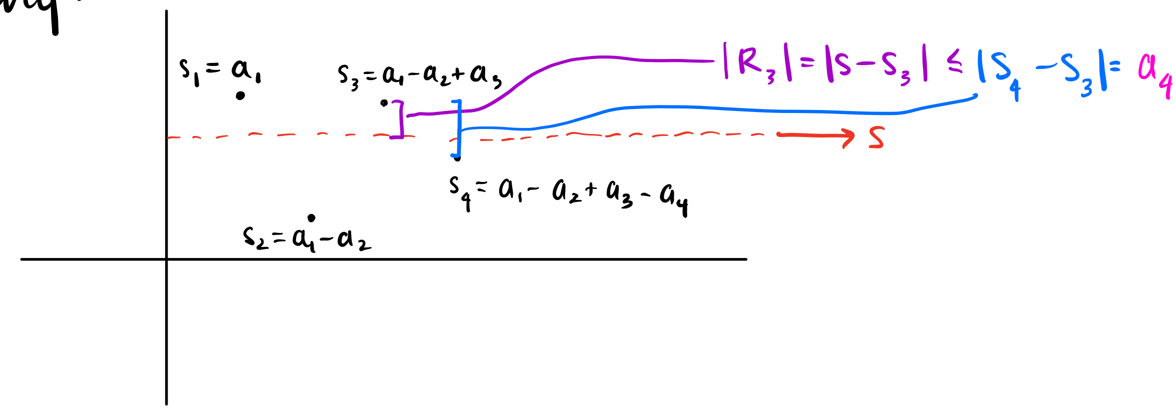
$R_n = a_{n+1} - a_{n+2} + a_{n+3} - \cdots$

or $-a_{n+1} + a_{n+2} - a_{n+3} + \cdots$

This says: "the error in estimating the sum by a partial sum is $\leq$ the next term"

Q. Why?

A.



ex/ $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = 1 - \frac{1}{10} + \frac{1}{100} - \frac{1}{1000} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{10^{n-1}}$

Q. How many terms must we add up to approximate within 0.001 the sum of this convergent series?

A. We want $|R_n| \le .001 = \frac{1}{1000}$, so need

$$|R_n| \le |a_{n+1}| \le \frac{1}{1000}$$

So use 3 terms!

$$S_3 = 1 - \frac{1}{10} + \frac{1}{100} = 1 - .1 + .01 = .91$$

and $S \approx .91$ with error $\le .001$

Absolute convergence

DEF. A series $\sum a$ is called **absolutely convergent** if $\sum |a_n|$ is convergent.

A series is called **conditionally convergent** if it is convergent but not absolutely convergent.

by AST

So ① $\sum (-1)^n a_n$ conv (by AST) $(a_n \geq 0)$

② $\sum a_n$ DIV

Absolute Convergence Theorem: If a series is absolutely convergent then it is convergent.

absolutely convergent series $\sum \frac{1}{n^p}$ for $p>1$ $\quad \sum ar^n$ for $|r|<1$

conditionally convergent $\sum \frac{(-1)^{n-1}}{n}$ $\quad \sum \frac{(-1)^{n-1}\ln(n)}{n}$

divergent series $\sum \frac{1}{n}$

$\sum \frac{1}{n^p}$ for $p \leq 1$ $\quad \sum ar^n$ for $|r| \geq 1$

ex1/ Q. Is $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n}$ ← a_n absolutely convergent? conditionally conv.?

or divergent?

A. First, consider $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{5^n}\right|$ ← $|a_n|$ $= \sum \frac{1}{5^n}$. This is conv. by GST

because $|r| = \frac{1}{5} < 1$

So $\sum \frac{(-1)^n}{5^n}$ is $\boxed{\text{absolutely convergent.}}$

(New and improved) Geometric series test (GST) $\sum_{n=1}^{\infty} ar^{n-1}$ converges absolutely if $|r| < 1$ and diverges if $|r| \geq 1$.

ex2/ Q. For what x is $\sum_{n=0}^{\infty} (-1)^n x^n$ absolutely convergent?

Conditionally conv.? Divergent?

A. $|x| < 1 \Rightarrow \boxed{\text{abs. conv.}}$

$|x| \geq 1 \Rightarrow \boxed{\text{divergent}}$

challenge: why true when $x = \pm 1$?

note: A geometric series is never conditionally convergent.

a_n

ex 3/ Q. Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n)}{n^2}$ absolutely convergent? conditionally conv.? or divergent?

A. $\displaystyle\sum_{n=1} \frac{(-1)^n \cos(n)}{n^2}$ has both pos. & neg. terms, but it's not alternating, so AST doesn't apply.

$|a_n|$

Still, $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cos(n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$ and, since $|\cos(n)| \le 1$,

$\displaystyle 0 \le \frac{|\cos(n)|}{n^2} \le \frac{1}{n^2}$, and $\displaystyle\sum \frac{1}{n^2}$ is a conv p-series $(p = 2 > 1)$.

So, by comparison test, $\displaystyle\sum \frac{|\cos(n)|}{n}$ is CONV as well

Thus $\displaystyle\sum \frac{(-1)^n \cos(n)}{n^2}$ is $\boxed{\text{absolutely convergent}}$

ex 4/ Q. Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ absolutely convergent? conditionally conv.? or divergent?

A. $\displaystyle\sum \left| \frac{(-1)^n}{n^5} \right| = \sum \frac{1}{n^5}$ conv. p-series so $\sum |a_n|$ conv

so $\sum a_n$ is $\boxed{\text{abs. conv.}}$

ex4/**Q**. Is $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ abs. conv.? cond. conv.? or div.?

A. ① Consider $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln(n)}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$, which is $\boxed{\text{divergent}}$ by CT with $\sum \frac{1}{n}$, the harmonic series

② So we use AST: ① $\frac{1}{\ln(n)} > \frac{1}{\ln(n+1)}$ ✓ & ② $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$ ✓

so the alternating series is $\boxed{\text{conv.}}$ by AST

③ So, $\sum a_n$ is $\boxed{\text{conditionally convergent}}$

Is $\sum a_n$ ABS CONV, COND CONV, OR DIV?

① $\sum |a_n| \longrightarrow$ CONV ✓ $\boxed{\sum a_n \text{ ABS CONV}}$

$\searrow$ DIV

② Use AST to test $\sum a_n \longrightarrow$ CONV ✓ $\boxed{\sum a_n \text{ COND CONV}}$

$\searrow$ doesn't work

③ test for DIV to show $\boxed{\sum a_n \text{ DIV}}$

TWO THINGS:

$\sum |a_n|$ DIV

$\sum a_n$ CONV by AST