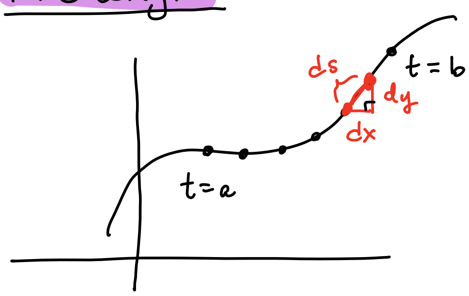
Parametric Integrals:

$$\begin{Bmatrix} x = f(t) \\ y = g(t) \end{Bmatrix} \quad a \le t \le b$$

① Arc length:



Pythagorean theorem ⇒

$$ds^2 = dx^2 + dy^2$$

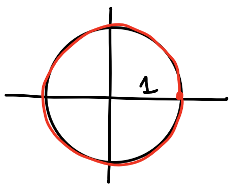
arc length differential $ds = \sqrt{dx^2 + dy^2}$

so $\boxed{ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$ in the parametric setting

$$= \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

So $AL = \int ds = \boxed{\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \text{arc length}}$ if the curve is traced out once for $a \le t \le b$

ex/ $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ $0 \le t \le 2\pi$



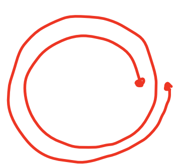
$$AL = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt$$

$\frac{dx}{dt} = -\sin(t)$ $\frac{dy}{dt} = \cos(t)$

$\boxed{\sin^2(t) + \cos^2(t) = 1}$

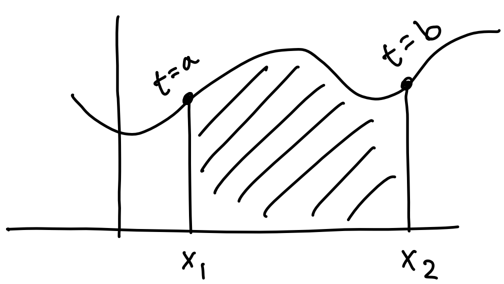
$$= \int_0^{2\pi} \sqrt{1}\, dt = \int_0^{2\pi} dt = t\Big]_0^{2\pi} = \boxed{2\pi}$$

<u>note</u>: $\int_0^{4\pi} = 4\pi$ curve is traced out twice!



② Area Under a Curve:

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

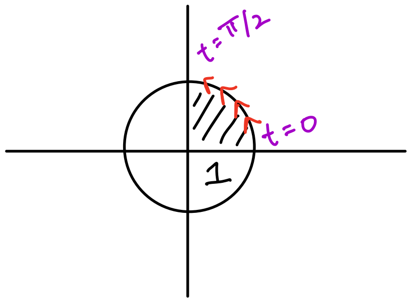


$y = g(t)$ $\frac{dx}{dt} = f'(t) \Rightarrow dx = f'(t)dt$

$$\text{area}^* = \left| \int_{x_1}^{x_2} y\, dx \right| = \left| \int_a^b g(t) f'(t)\, dt \right|$$

*provided the curve is always above or always below x-axis ex/ $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ Compute the area inside the cure.

$\Rightarrow dx = -\sin(t)\,dt$



$$A = 4\left|\int_0^{\pi/2} \sin(t)(-\sin(t))\,dt\right|$$

$$= 4\left|-\int_0^{\pi/2} \sin^2(t)\,dt\right|$$

$$\boxed{\sin^2(t) = \frac{1-\cos(2t)}{2}}$$

$$\boxed{\cos^2(t) = \frac{1+\cos(2t)}{2}}$$

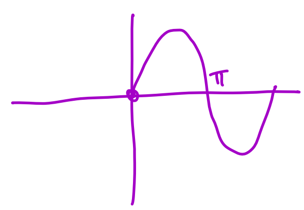
$$= 4\left|-\int_0^{\pi/2} \frac{1-\cos(2t)}{2}\,dt\right|$$

$$= 4\left|-\int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)dt\right|$$

$\frac{d}{dt}\sin(2t) = 2\cos(2t)$

$$= 4\left|-\left[\frac{1}{2}t - \frac{1}{4}\sin(2t)\right]_0^{\pi/2}\right|$$

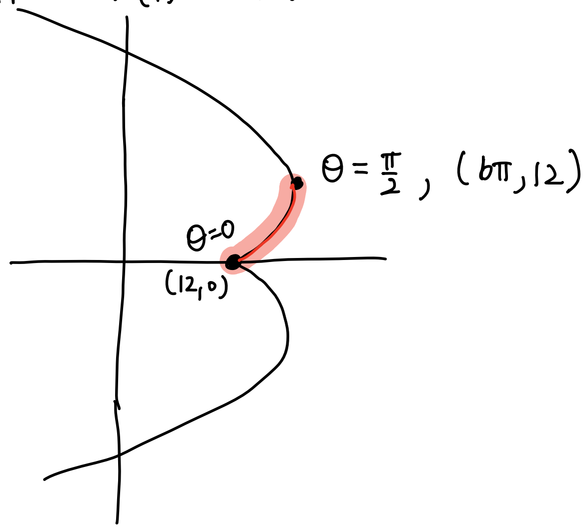
$\sin\left(2\left(\frac{\pi}{2}\right)\right) = \sin(\pi) = 0$



$$= 4\left|-\left(\frac{\pi}{4} - 0\right)\right|$$

$$= 4\left|-\frac{\pi}{4}\right| = 4\left(\frac{\pi}{4}\right) = \boxed{\pi}$$

area $= \pi r^2 = \pi(1)^2 = \pi$ ✓



Examples:

θ is the parameter

① $\begin{Bmatrix} x = 12(\cos\theta + \theta\sin\theta) \\ y = 12(\sin\theta - \theta\cos\theta) \end{Bmatrix}$

at $\theta = 0$, $(x, y) = (12, 0)$

at $\theta = \frac{\pi}{2}$, $(x, y) = (\underbrace{12 \cdot \frac{\pi}{2}}_{\approx 19}, 12)$

What is the length of the curve for $\theta = 0$ to $\theta = 12$?

$$AL = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

prod. rule

① $\frac{dx}{d\theta} = 12\left(\cancel{-\sin\theta} + \cancel{\sin\theta} + \theta\cos\theta\right) = 12\theta\cos\theta$

$\frac{dy}{d\theta} = 12\left(\cancel{\cos\theta} - \cancel{\cos\theta} + \theta\sin\theta\right) = 12\theta\sin\theta$

② $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (12\theta\cos\theta)^2 + (2\theta\sin\theta)^2$

$= 12^2\theta^2(\cos^2\theta + \sin^2\theta)$

$= 12^2\theta^2 = 144\theta^2$

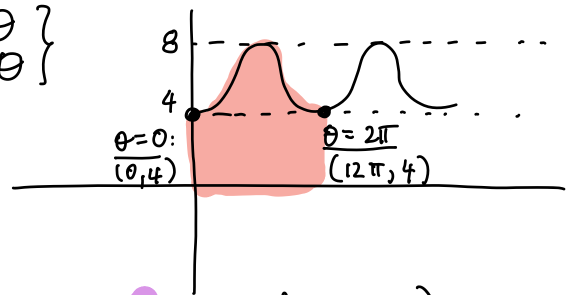
③ $\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{144\theta^2} = 12|\theta| = \boxed{12\theta}$ (since $\theta > 0$)

(4) So $AL = \displaystyle\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/2} 12\theta\, d\theta = 6\theta^2 \Big]_0^{\pi/2} = 6\left(\frac{\pi}{2}\right)^2$

$= \dfrac{3\pi^2}{2} \approx 14.804\ldots$

(2) Find the area under one arch of the trochoid

$$\begin{cases} x = 6\theta - 2\sin\theta \\ y = 6 - 2\cos\theta \end{cases}$$



(a) find endpoints of integration:

at $\theta = 0$: $(0, 4)$

the next min occurs when $\theta = 2\pi$, or $(12\pi, 4)$

(b) compute $dx = (6 - 2\cos\theta)\, d\theta$

$(a-b)^2 = a^2 - 2ab + b^2$

(c) $A = \displaystyle\int_0^{2\pi} y\,dx = \int_0^{2\pi} (6 - 2\cos\theta)(6 - 2\cos\theta)\, d\theta$

$\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$

$= \displaystyle\int_0^{2\pi} (36 - 24\cos\theta + 4\cos^2\theta)\, d\theta$

$\Rightarrow 4\cos^2\theta = 2(1 + \cos 2\theta)$

$= 2 + 2\cos 2\theta$

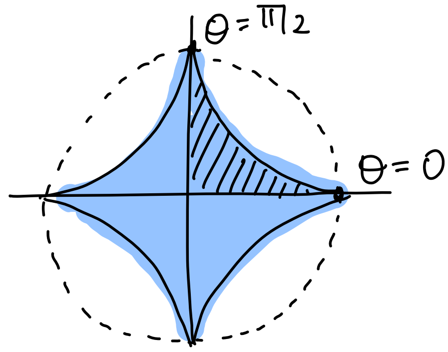
$= \displaystyle\int_0^{2\pi} (36 - 24\cos\theta + 2 + 2\cos 2\theta)\, d\theta$

$= \displaystyle\int_0^{2\pi} (38 - 24\cos\theta + 2\cos(2\theta))\, d\theta$

$= 38\theta - \cancel{24\sin\theta}^{\,0} + \cancel{\sin(2\theta)}^{\,0} \Big|_0^{2\pi} = 38(2\pi) \approx 238.76104..$

check $\frac{d}{d\theta}\sin(2\theta) = 2\cos(2\theta)$

3 Find the area inside the asteroid $\begin{cases} x = 4\cos^3\theta \\ y = 4\sin^3\theta \end{cases}$ $0 \le \theta \le 2\pi$



1 pick endpoints: $A = 4\int_0^{\pi/2} y\,dx \quad = 2\int_0^{\pi} y\,dx$

2 compute $dx = 4 \cdot 3\cos^2\theta \cdot (-\sin\theta)\,d\theta$

$$= -12\cos^2\theta\sin\theta\,d\theta$$

3 $A = 4\int_0^{\pi/2} 4\sin^3\theta\left(-12\cos^2\theta\sin\theta\right)d\theta$

$$= -4^2 \cdot 12\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta$$

$$= -4^{\cancel{2}} \cdot 12\int_0^{\pi/2} \cancel{\tfrac{1}{4}}\left[1 - 2\cos 2\theta + \cos^2 2\theta\right)\cos^2\theta\,d\theta$$

$$= -\overset{2}{\cancel{4}} \cdot 12\int_0^{\pi/2} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right)\frac{(1+\cos 2\theta)}{\cancel{2}}\,d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\Rightarrow \sin^4\theta = \left(\frac{1-\cos 2\theta}{2}\right)^2$

$= \frac{1}{4}\left(1 - 2\cos 2\theta + \cos^2 2\theta\right)$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= -24\int_0^{\pi/2} \left(1 - 2\cos(2\theta) + \cos^2(2\theta) + \cos(2\theta) - 2\cos^2(2\theta) + \cos^3(2\theta)\right) d\theta$$

$$= -24\int_0^{\pi/2} (1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta)\, d\theta$$

$$\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$$

$$= -24 \int_0^{\pi/2} \left(1 - \cos 2\theta - \left(\frac{1+\cos 4\theta}{2}\right) + (1-\sin^2 2\theta)\cos 2\theta\right) d\theta$$

$$\cos^2(2\theta) = 1 - \sin^2(2\theta)$$

$$= -24\left[\theta - \cancelto{0}{\tfrac{1}{2}\sin(2\theta)} - \tfrac{1}{2}\left(\theta + \cancelto{0}{\tfrac{1}{4}\sin(4\theta)}\right) + \cancelto{0}{\tfrac{1}{2}\sin 2\theta} - \cancelto{0}{\tfrac{1}{6}\sin^3(2\theta)}\right]_0^{\pi/2}$$

$$= -24\left(\tfrac{1}{2}\theta\right)_0^{\pi/2} = -12\left(\pi/2\right) = -6\pi$$

so area $= |-6\pi| = 6\pi \approx 18.8495\ldots$