Math 143: Calculus III

Midterm II

March 29, 2022

Name: Solutions (Please print clearly)

UR ID: ______________________

Indicate the lecture time you are registered for with a check in the appropriate box:

☐ MW 10:25–11:40am (Cook)

☐ TR 2:00-3:15pm (Sahay)

Instructions:

- You have 75 minutes to work on this exam. You are responsible for checking that this exam has all 10 pages. **Please do not remove any pages.**
- Write your final answers in the provided answer boxes.
- No calculators, phones, electronic devices, books, or notes are allowed during the exam, except for the provided formula sheet.
- Show all work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required, **except when specifically stated otherwise.**
- Please write your UR ID in the space provided at the top of each page.

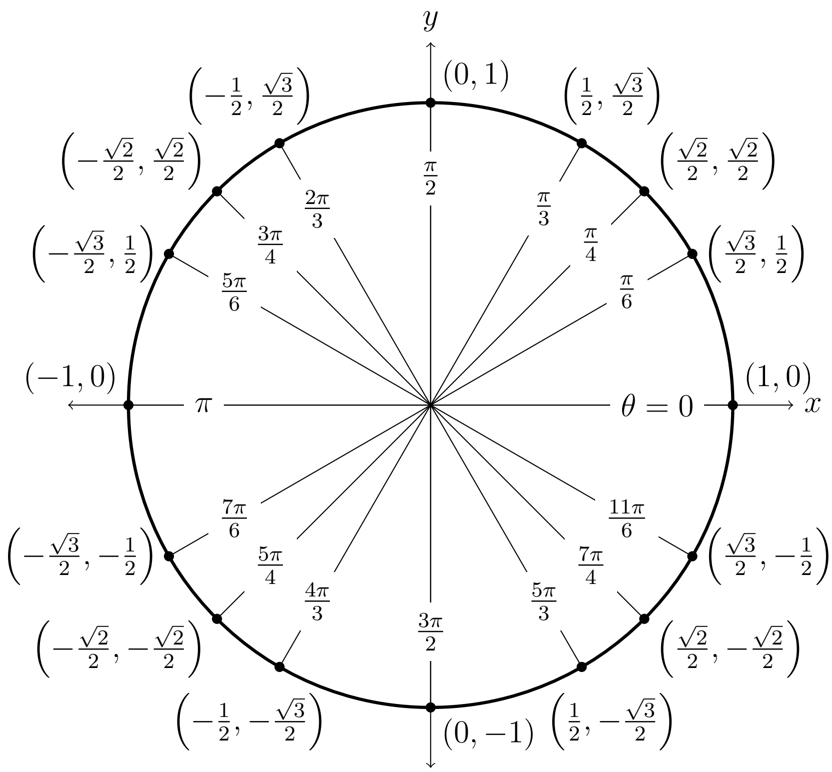
Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________

FOR REFERENCE, NO QUESTION ON THIS PAGE

Unit circle: The coordinates of the endpoints satisfy $(x, y) = (\cos\theta, \sin\theta)$, where θ is the corresponding angle.



Common Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad R + 1$$

1. (15 points)

For the following series, determine whether the series converges absolutely, converges conditionally, or diverges. You may use whichever test is appropriate. **Clearly state any test you use, and make sure to check that the hypothesis of the test is satisfied.**

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\,(n^2+2)$

$\lim_{n\to\infty}(n^2+2)=\infty$

$\lim_{n\to\infty}-(n^2+2)=\infty$

therefore, $\lim_{n\to\infty}(-1)^n(n^2+2)$ DNE, so by the divergence test, the series diverges

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{n(n+1)(n+2)}{3^n}$

ratio test:

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{(n+1)(n+2)(n+3)3^n}{n(n+1)(n+2)3^{n+1}}=\lim_{n\to\infty}\frac{(n+3)}{3n}$$

$$=\frac{1}{3}<1$$

series is absolutely convergent by the ratio test

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{n^2+1}{n+1}\right)^n$

root test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{n^2+1}{n+1}=\infty$$

series diverges by the root test

2. (15 points)

For the following series, determine whether the series converges absolutely, converges conditionally, or diverges. You may use whichever test is appropriate. **Clearly state any test you use, and make sure to check that the hypothesis of the test is satisfied.**

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ → this is an alternating series

let $b_n = |a_n| = \frac{1}{2n+1}$.

- $\frac{1}{2(n+1)+1} < \frac{1}{2n+1}$
- $\lim_{n\to\infty} \frac{1}{2n+1}$

by the Alternating series test, the series converges

↳ $\boxed{\text{the series is conditionally convergent}}$

$\sum_{n=1}^{\infty} \frac{1}{2n+1}$ is divergent by LCT:

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2n+1}} = \lim_{n\to\infty} \frac{2n+1}{n} = 2$$

so, the series is not absolutely convergent

(b) $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!\,3^n}{n!\,3^{n+1}} = \lim_{n\to\infty} \frac{n+1}{3} = \infty$$

By the ratio test, the series diverges

3. (20 points)
For the following power series, determine the interval of convergence using the ratio or the root test.

(a) $\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!}$

ratio test:

$$\lim_{n\to\infty} \frac{|x|^{n+1}(2n+1)!}{|x|^n(2n+3)!} = |x| \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0 \quad \text{for all } x$$

the interval of convergence is $(-\infty, \infty)$

(b) $\sum_{n=0}^{\infty} n! x^n$

ratio test:

$$\lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{n!|x|^n} = |x| \lim_{n\to\infty} (n+1) = \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

the interval of convergence is $\{0\}$

(continued)

For the following power series, determine the interval of convergence using the ratio or the root test.

(c) $\sum_{n=0}^{\infty}(-1)^n \frac{(x-2)^n}{2^n(2n+1)}$

$$\lim_{n\to\infty} \frac{(|x-2|)^{n+1} 2^n (2n+3)}{(|x-2|)^n 2^{n+1}(2n+1)} = |x-2| \lim_{n\to\infty} \frac{2n+3}{2(2n+1)}$$

$$= \frac{1}{2}|x-2|$$

converges for $|x-2|<2$, i.e., $x \in (0,4)$

check endpoints:

$x=0$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-2)^n}{2^n(2n+1)} = \sum_{n=0}^{\infty} \frac{1}{2n+1}, \text{ diverge}$$

(see 2a)

$x=4$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{2^n(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}, \text{ converge}$$

(see 2a)

$\boxed{\text{the interval of convergence is } (0,4]}$

4. (10 points) Express the following function as a power series and find its radius of convergence.

$$f(x) = \frac{x}{2 + x^2}$$

$$f(x) = \frac{x}{2}\left(\frac{1}{1-\left(-\frac{x^2}{2}\right)}\right) = \frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^{n+1}}$$

converges for $\left|\frac{-x^2}{2}\right| < 1$

$\hookrightarrow R = \sqrt{2}$

5. (10 points) Express the following function as a power series and find its radius of convergence.

$$f(x) = \frac{1}{(2-x)^2}$$

$$f(x) = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\frac{1}{2}\cdot\frac{1}{1-\left(\frac{x}{2}\right)}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{n\,x^{n-1}}{2^n}$$

$$R = 2$$

6. (15 points) For each of the following functions, find the Maclaurin series and its radius of convergence.

(a) $x \cos x$ $= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$,

$R = \infty$

(b) $\ln(1 + 2x)$ $= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2x)^n}{n}$, $R = \frac{1}{2}$

7. (15 points) Determine the function each of the following Maclaurin series represents.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}$

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} = x\cos x$