Math 143: Calculus III

Midterm I

February 17, 2022

Name: solutions (Please print clearly)

UR ID: ____________________

Indicate the lecture time you are registered for with a check in the appropriate box:

☐ MW 10:25–11:40am (Cook)

☐ TR 2:00-3:15pm (Sahay)

Instructions:

- You have 180 minutes to work on this exam. You are responsible for checking that this exam has all 10 pages. **Please do not remove any pages.**
- Write your final answers in the provided answer boxes.
- No calculators, phones, electronic devices, books, or notes are allowed during the exam, except for the provided formula sheet.
- Show all work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required, **except when specifically stated otherwise.**
- Please write your UR ID in the space provided at the top of each page.

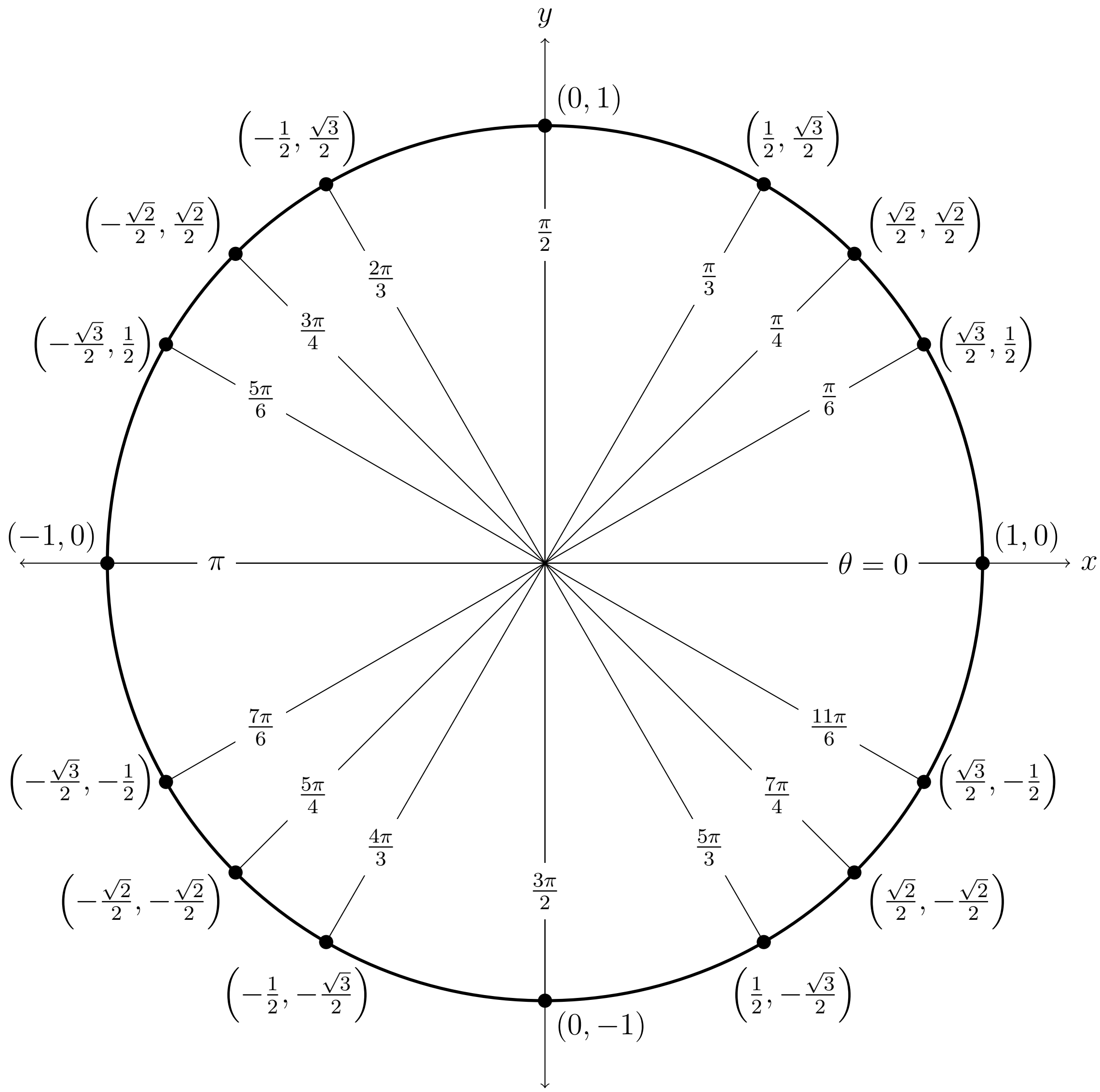
Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________________

FOR REFERENCE, NO QUESTION ON THIS PAGE

Unit circle: The coordinates of the endpoints satisfy $(x, y) = (\cos\theta, \sin\theta)$, where θ is the corresponding angle.



1. (15 points) Determine whether the following **sequences** converge or diverge. If they converge, find their limit. If they diverge, determine whether they diverge to $+\infty$, $-\infty$, or because they oscillate.

(a)

$$\left\{\frac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$$

$$-\frac{1}{n^2} \leq \frac{(-1)^n}{n^2} \leq \frac{1}{n^2}$$

$\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$ by the Squeeze theorem

(b)

$$\{(-n)^2\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} (-n)^2 = \lim_{n\to\infty} n^2 = \infty$$

sequence diverges to ∞

(c)

$$\left\{\frac{(-1)^n(n+1)^2}{n}\right\}_{n=1}^{\infty}$$

for n even, terms approach ∞

for n odd, terms approach $-\infty$

the sequence diverges because it oscillates

2. (10 points) Determine whether the following **sequences** converge or diverge. If they converge, find their limit. If they diverge, determine whether they diverge to $+\infty$, $-\infty$, or because they oscillate.

(a)

$$\{\cos(n\pi)\}_{n=1}^{\infty}$$

for n even: $\cos(n\pi) = 1$
for n odd: $\cos(n\pi) = -1$

sequence diverges because it oscillates

(b)

$$\left\{\frac{\cos(n\pi)}{2^n}\right\}_{n=1}^{\infty}$$

$$-\frac{1}{2^n} \leq \frac{\cos(n\pi)}{2^n} \leq \frac{1}{2^n}$$

by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{\cos(n\pi)}{2^n} = 0$$

3. (10 points) Determine whether the following **series** converge or diverge. If they converge, find their sum.

(a)

$$\sum_{n=1}^{\infty} \frac{5^{2n}}{6^{n-1}}$$

$$= \sum_{n=1}^{\infty} 25 \cdot \frac{(25)^{n-1}}{6^{n-1}}$$

geometric series with $r = \frac{25}{6}$,

divergent

(b)

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{\pi^n}$$

$$= \sum_{n=1}^{\infty} 2 \cdot \left(\frac{2}{\pi}\right)^{n-1}$$

geometric series, $r = \frac{2}{\pi}$

$|r| < 1$

series converges to $\frac{2}{1 - 2/\pi}$

4. (15 points) Determine whether the following **series** converge or diverge. **Show all work and name any test you use.**

(a)

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

diverges by the divergence test

(b)

$$\sum_{n=1}^{\infty} \frac{n^3-1}{n^5+1}$$

$$\frac{n^3-1}{n^5+1} \leq \frac{n^3}{n^5+1} \leq \frac{n^3}{n^5} = \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so by the direct comparison test, the series converges.

5. (10 points) Use the **integral test** to determine if the following series converges or diverges. You must use the integral test to get full credit.
Hint: $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$f(x) = \frac{\sqrt{x}}{1+x^{3/2}}$

is positive, decreasing, continuous on $[1,\infty)$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$$

$$\int_1^\infty \frac{\sqrt{x}}{1+x^{3/2}}\,dx = \int_2^\infty \frac{2}{3}\cdot\frac{1}{u}\,du$$

$u = 1+x^{3/2}$

$du = \frac{3}{2}x^{1/2}dx$

$$= \lim_{t\to\infty}\left(\frac{2}{3}\ln(t) - \frac{2}{3}\ln(2)\right)$$

$$= \infty.$$

The integral is divergent, so by the integral test, the series is divergent

6. (10 points) Use the **direct comparison test** to determine if the following series converge or diverge by comparing it to a geometric series or p-series. You must use the direct comparison test to get full credit.

$$\sum_{n=1}^{\infty} \frac{2^n}{(3n)^n}$$

$\frac{2^n}{(3n)^n} \leq \frac{2^n}{3^n}$, $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series

$\hookrightarrow$ by DCT, the series converges

7. (15 points) Consider the following sequence, assuming that the pattern continues:

$$\{a_n\} = \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \dots \right\}$$

(a) Find an equation for the nth term a_n.

$$a_n = \frac{n}{2^n}$$

(b) Evaluate $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} \frac{n}{2^n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{2^n \ln(2)} = 0$$

(c) Is the **series** $\sum_{n=1}^{\infty} a_n$ a geometric series?

NO: look at the ratio $\frac{a_{n+1}}{a_n} = \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n}$

$$= \frac{n+1}{2n}$$

the common ratio is not constant

8. (15 points) Suppose a series $\sum_{n=1}^{\infty} a_n$ has partial sums given by

$$S_N = \frac{N}{N+1}.$$

(a) What is $\sum_{n=1}^{5} a_n$?

$$\sum_{n=1}^{5} a_n = S_5 = \frac{5}{6}$$

(b) What is the value of a_5?

$$a_5 = S_5 - S_4 = \frac{5}{6} - \frac{4}{5}$$

$$= \frac{25}{30} - \frac{24}{30} = \frac{1}{30}$$

(c) Does $\sum_{n=1}^{\infty} a_n$ converge? If so, what is its sum?

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 1$$

The series converges to 1.