Math 143: Calculus III

Final Exam

May 2, 2022

Name: ______________________________(Please print clearly)

UR ID: ______________________________

Indicate the lecture time you are registered for with a check in the appropriate box:

☐ MW 10:25–11:40am (Cook)

☐ TR 2:00-3:15pm (Sahay)

Instructions:

- You have 180 minutes to work on this exam. You are responsible for checking that this exam has all 15 pages. **Please do not remove any pages.**
- Write your final answers in the provided answer boxes.
- No calculators, phones, electronic devices, books, or notes are allowed during the exam, except for the provided formula sheet.
- Show all work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required, **except when specifically stated otherwise.**
- Please write your UR ID in the space provided at the top of each page.

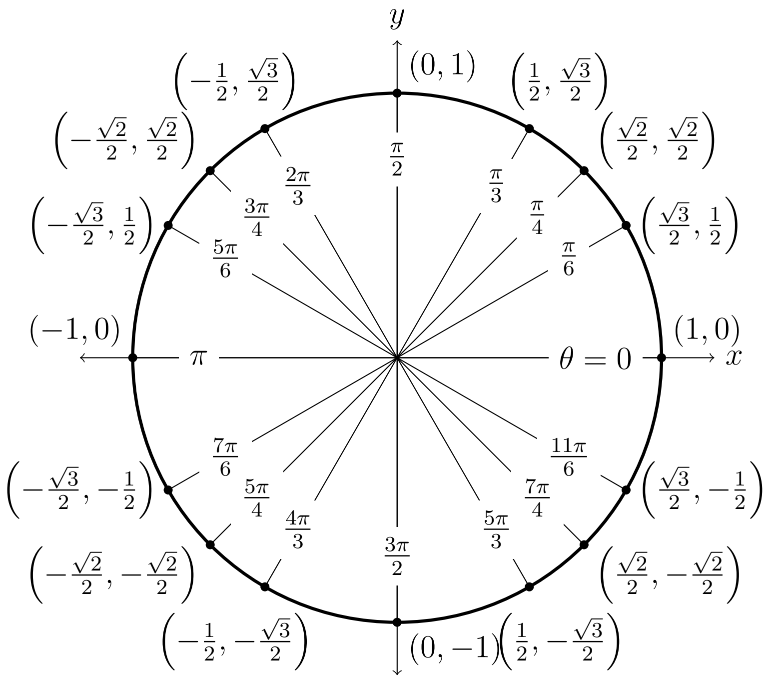
Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

FOR REFERENCE, NO QUESTION ON THIS PAGE

Unit circle: The coordinates of the endpoints satisfy $(x, y) = (\cos\theta, \sin\theta)$, where θ is the corresponding angle.



Common Maclaurin series:

$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n,$	$R = 1$
$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!},$	$R = \infty$
$\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$	$R = \infty$
$\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$	$R = \infty$
$\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$	$R = 1$
$\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n},$	$R + 1$

Formulas for a parametric curve:

$x = f(t), y = g(t)$

Arc length from $t = a$ to $t = b$:

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

Area under the curve from $t = a$ to $t = b$

$$\int_a^b g(t) f'(t)\,dt \qquad \left[\text{ or } \int_b^a g(t) f'(t)\,dt\right]$$

Formulas for a polar curve:

$r = f(\theta)$

Arc length from $\theta = a$ to $\theta = b$:

$$\int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta$$

Area bounded by the curve and the rays $\theta = a, \theta = b$:

$$\int_a^b \frac{1}{2} (f(\theta))^2\,d\theta$$

PART I

1. (15 points) Find the limit of the **sequence** or show that the limit does not exist.

(a)

$$\{\cos(\pi n)\}_{n=1}^{\infty}$$

for n even: $\lim_{n\to\infty} \cos(\pi n) = 1$

for n odd: $\lim_{n\to\infty} \cos(\pi n) = -1$

therefore the limit does not exist

(b)

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

(c)

$$\left\{\frac{n+1}{n-1}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} \frac{n+1}{n-1} = \lim_{n\to\infty} \frac{1+n^{-1}}{1-n^{-1}} = 1$$

PART I

2. (10 points) Use the **direct comparison test** or the **limit comparison test** to determine whether the series converges or diverges. To receive full credit, you must clearly state the series to which you are comparing the given series.

(a)

$$\sum_{n=1}^{\infty} \frac{n^7+n}{(n^4+n)^2}$$

compare to: $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent p-series)

$$\lim_{n\to\infty} \frac{\frac{n^7+n}{(n^4+n)^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^8+n^2}{n^8+2n^5+n^2} = 1$$

By the limit comparison test, the series diverges

(b)

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n-1}$$

compare to: $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ (divergent geometric series)

$$\lim_{n\to\infty} \frac{\frac{3^{n-1}}{2^n-1}}{\frac{3^n}{2^n}} = \lim_{n\to\infty} \frac{3^{n-1}\cdot 2^n}{2^{n-1}\cdot 3^n} = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3}$$

the series diverges by the limit comparison test

PART I

3. (15 points) Use the **ratio test** or the **root test** to determine whether the series converges or diverges.

(a)

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

ratio test:

$$\lim_{n\to\infty} \frac{(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = 0$$

series converges by the ratio test

(b)

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2-1}\right)^n$$

root test:

$$\lim_{n\to\infty} \frac{n+1}{n^2-1} = 0$$

series converges by the root test

PART I

4. (15 points) Determine the interval of convergence for the power series.

(a)

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n!}$$

apply ratio test:

$$\lim_{n\to\infty} \frac{|x|^{n+2}}{(n+1)!} \cdot \frac{n!}{|x|^{n+1}} = |x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 \quad \text{(for all } x\text{)}$$

the interval of convergence is $\underline{R=\infty}$

(b)

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2+1}$$

apply ratio test:

$$\lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{|x-1|^n} = |x-1| \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2}$$

$$= |x-1|,$$

series converges when $|x-1|<1$,

the radius of convergence is $\underline{R=1}$

$x=0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

both converge by LCT (compare to $\sum \frac{1}{n^2}$)

$x=2$: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\boxed{\text{IOC: } [0,2]}$

PART I

5. (15 points)

Find a power series expansion centered at 0 for the following functions. *Hint: recall the expansion for* $\frac{1}{1-x}$.

(a) $\frac{1+x}{1+x^2}$ $= (1+x)\left(\frac{1}{1-(-x^2)}\right) = (1+x)\sum_{n=0}^{\infty}(-x^2)^n$

(b) $\frac{3x^2}{(1-x^3)^2}$ $\frac{d}{dx}\left(\frac{1}{1-x^3}\right) = \frac{3x^2}{(1-x^3)^2}$

$\parallel$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-x^3)^n\right) = \sum_{n=0}^{\infty}\frac{d}{dx}(-x^{3n})$$

$$\boxed{= \sum_{n=1}^{\infty}(-1)^{3n}\cdot 3n\cdot x^{3n-1}}$$

PART I

6. (15 points) Compute the following limits. (*Hint: Maclaurin series*)

(a) $\lim_{x\to 0} \frac{\ln(1+4x) - 4x + 2x^2}{x^3}$

$$= \lim_{x\to 0} \frac{\left(\sum_{n=1}^{\infty} \frac{(4x)^n}{n} \cdot (-1)^{n-1}\right) - 4x + 2x^2}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{x^3} \cdot (-8x^2 + 2x^2) + \frac{1}{x^3} \cdot \frac{4^3 x^3}{3} + \dots$$

$$= \lim_{x\to 0} -\frac{6}{x} \to -\infty$$

(b) $\lim_{x\to 0} \frac{\sin(2x) - 2x\cos x}{x^2}$ $= \lim_{x\to 0} \frac{\left(\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}\right) - 2x\left(\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}\right)}{x^2}$

$$= \lim_{x\to 0} \frac{\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots\right) - \left(2x \cdot 1 - 2x \cdot \frac{x^2}{2!} + \dots\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{-\frac{2^3 x^3}{3!} + \frac{2x^3}{2!} + \frac{(2x)^5}{5!} + \dots}{x^2}$$

$$= 0$$

PART I

7. (15 points)

(a) Find the Maclaurin series for $f(x) = \sin(\frac{\pi}{2} + x)$ from first principles (in other words, use the definition of a Taylor series).

$f^{(1)}(x) = \cos(\frac{\pi}{2}+x)$

$f^{(2)}(x) = -\sin(\frac{\pi}{2}+x)$

$f^{(3)}(x) = -\cos(\frac{\pi}{2}+x)$

$f^{(4)}(x) = \sin(\frac{\pi}{2}+x)$

$\vdots$

etc

$f^{(0)}(0) = 1$

$f^{(1)}(0) = 0$

$f^{(2)}(0) = -1$

$f^{(3)}(0) = 0$

$f^{(4)}(0) = 1$

$\vdots$

Maclaurin series:

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

(b) Compare this power series to the known Maclaurin series for $\cos x$. Can you conclude a relationship between $\sin(\frac{\pi}{2} + x)$ and $\cos x$?

The series we found in part (a) is exactly the Maclaurin series for $\cos x$.

So, $\cos x = \sin(\frac{\pi}{2} + x)$.

PART II

8. (20 points) Consider the parametric curve given by

$$x = t^3, \qquad y = t^2 - 2t.$$

(a) Find a function f so that $y = f(x)$ represents the same curve as the above paratrization.

$t = x^{1/3}, \qquad y = (x^{1/3})^2 - 2x^{1/3}$

(b) At what points (x, y) is the tangent line to the curve horizontal?

$\frac{dy}{dt} = 2t - 2 = 0 \longrightarrow t = 1$ horizontal: $(1, -1)$

$\frac{dx}{dt} = 3t^2 = 0 \longrightarrow t = 0$ vertical: $(0, 0)$

(c) At what points (x, y) is the tangent line to the curve vertical?

(d) Find the equation of the tangent line at $(x, y) = (1, -1)$.

$y + 1 = 0(x - 1)$

$\hookrightarrow y = -1$

PART II

9. (20 points) Consider the parametric curve given by

$$x = \cos^2 t, \qquad y = \sin^3 t$$

(a) Find a function F so that $F(x, y) = 0$ represents the same curve as the above paratrization.

$$x^2 + y^{2/3} = \cos^2 t + \sin^2 t = 1$$

$$\hookrightarrow \quad F(x,y) = x^2 + y^{2/3} - 1$$

(b) Write an expression that represents the the area enclosed by x-axis, the positive y-axis, and this part of the curve. *Hint: try sketching the curve for* $0 \leq t \leq \pi/2$.

$$A = \int_{\pi/2}^{0} \sin^3 t \cdot (-2 \sin t \cos t)\, dt$$

(c) Evaluate the integral from part (b). (*Hint: u-substitution*)

$$A = 2 \int_0^{\pi/2} \sin^4 t \cos t \, dt$$

$$= 2 \left[\frac{1}{5} \sin^5 t \right]_0^{\pi/2} = \boxed{\frac{2}{5}}$$

PART II

10. (10 points)

(a) The point $(1, \sqrt{3})$ is given in Cartesian coordiantes. Find a representation of this point in polar coordinates.

$$r = \sqrt{1+3} = \pm 2,$$

$$\tan\theta = \sqrt{3}$$

$$\boxed{(r,\theta) = (2, \frac{\pi}{3})}$$

(b) The point $(1, \pi/2)$ is given in polar coordinates. Find the representation of this point in Cartesian coordinates.

$$x = r\cos\theta = 1 \cdot 0$$

$$y = r\sin\theta = 1 \cdot 1$$

$$\boxed{(x,y) = (0,1)}$$

PART II

11. (20 points) Consider the polar curve $r = \cos(2\theta)$ (a four-leaf rose).

(a) Find the values of θ for $0 \leq \theta \leq 2\pi$ where the curve passes through the origin.

$$r = \cos(2\theta) = 0 \longrightarrow 2\theta = \frac{\pi}{2} + n\pi$$

$$\rightarrow \theta = \frac{\pi}{4} + \frac{n\pi}{2}$$

values w/ $0 \leq \theta \leq 2\pi$:

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

(b) Set up, but do not evaluate, an integral that represents the area of one leaf of the rose.

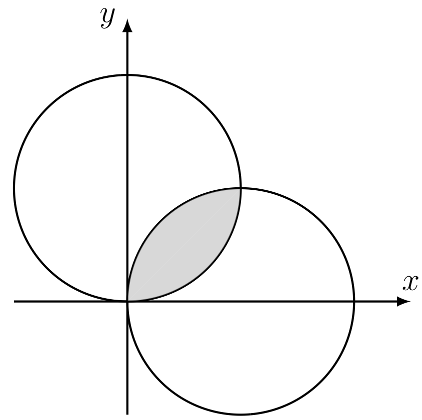
$$A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} (\cos(2\theta))^2 \, d\theta$$

(c) Set up, but do not evaluate, an integral that represents the arc length of one leaf of the rose.

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2(2\theta) + 4\sin^2(2\theta)} \, d\theta$$

PART II

12. (15 points) Consider the polar curves $r = \sin(\theta)$ and $r = \cos(\theta)$, which are plotted below.



(a) Find all points where the two curves intersect. Express these points in Cartesian coordinates, and clearly mark your answers.

$\sin\theta = \cos\theta$: $\theta = \frac{\pi}{4}$, $r = \frac{\sqrt{2}}{2}$

$$\boxed{(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)}$$

the curves also intersect at the origin:

$$\boxed{(x,y) = (0,0)}$$

(b) Set up, but do not evaluate, an integral (or sum of integrals) which represents the area of the shaded region.

$$A = \frac{1}{2}\int_0^{\pi/4} \sin^2\theta \, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta \, d\theta$$

PART II

13. (15 points) Consider the polar curve $r = 1 + \cos(\theta)$ (a cardioid).

(a) Set up, but do not evaluate, an integral that represents the area enclosed by the curve.

$$A = \frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2 \, d\theta$$

(b) Evaluate the integral from part (a). You may use the identity: $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$.

$$A = \frac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \frac{1}{2}(1 + \cos(2\theta)) \, d\theta$$

$$= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi + \pi) = \boxed{\frac{3\pi}{2}}$$