Math 143

Midterm 2 Solutions

April 5, 2018

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Circle your instructor's name:

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- **No calculators, notes, or other aids are allowed during this exam.**
- **Show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **You are responsible for checking this exam has all 8 pages.**
- **If possible evaluate trigonometric and logarithmic expressions. Otherwise you do not need to simplify.**

Please copy and sign the following statement.

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

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Signature: ____________________

1. (14 points) Use the Ratio or Root Test to determine whether each of these series converges to a finite value or diverges to ∞.

(a)

$$\sum_{n=1}^{\infty} \frac{(n+1)(7^2-1)^n}{7^{2n}}$$

Answer:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+2)(7^2-1)^{n+1}/7^{2(n+1)}}{(n+1)(7^2-1)^n/7^{2n}}\right| \\
&= \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{48^{n+1}}{48^n}\cdot\frac{49^n}{49^{n+1}} \\
&= \lim_{n\to\infty}\frac{48}{49}\cdot\frac{n+2}{n+1} = \frac{48}{49} < 1.
\end{aligned}$$

Since the limit is less than one, it converges by the ratio test.

(b)

$$\sum_{n=1}^{\infty} \frac{(n+5)^n}{e^{n^2}}$$

Answer:

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{a_n} &= \lim_{n\to\infty}\left(\frac{(n+5)^n}{e^{n^2}}\right)^{1/n} \\
&= \lim_{n\to\infty}\frac{n+5}{e^n} \\
&= \lim_{n\to\infty}\frac{1}{e^n} = 0 < 1.
\end{aligned}$$

Since the limit is less than one, it converges by the root test. Note the bottom line follows from L'Hospital's rule.

2. (15 points) Find the radius and interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(-8)^n (x-3)^{n+1}}{n5^n}$$

Answer:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-8)^{n+1}(x-3)^{n+2}/(n+1)5^{n+1}}{(-8)^n (x-3)^{n+1}/n5^n} \right| \\
&= \lim_{n\to\infty} \frac{8^{n+1}}{8^n} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{n}{n+1} \cdot \frac{|x-3|^{n+2}}{|x-3|^{n+1}} \\
&= \lim_{n\to\infty} \frac{8n}{5n+5} |x-3| \\
&= \frac{8}{5}|x-3|.
\end{aligned}$$

This is less than 1 whenever $|x-3| < \frac{5}{8}$, so the radius of convergence is $R = \frac{5}{8}$. To determine the interval of convergence we must test the endpoints: $x = 3 - \frac{5}{8}$ and $x = 3 + \frac{5}{8}$.

$$\begin{aligned}
x = 3 - \frac{5}{8}: \quad \sum_{n=1}^{\infty} \frac{(-8)^n((3-5/8)-3)^{n+1}}{n5^n} &= \sum_{n=1}^{\infty} \frac{(-8)^n(-5/8)^{n+1}}{n5^n} \\
&= \sum_{n=1}^{\infty} \frac{-5^{n+1}}{n \cdot 8 \cdot 5^n} \\
&= -\frac{5}{8} \sum_{n=1}^{\infty} \frac{1}{n}.
\end{aligned}$$

$$\begin{aligned}
x = 3 + \frac{5}{8}: \quad \sum_{n=1}^{\infty} \frac{(-8)^n((3+5/8)-3)^{n+1}}{n5^n} &= \sum_{n=1}^{\infty} \frac{(-8)^n(5/8)^{n+1}}{n5^n} \\
&= \sum_{n=1}^{\infty} (-1)^n \frac{5^{n+1}}{n \cdot 8 \cdot 5^n} \\
&= \frac{5}{8} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.
\end{aligned}$$

In the first case we have the harmonic series, which diverges. In the second case we have the alternating harmonic series, which converges. So the interval of convergence is $I = (3 - \frac{5}{8}, 3 + \frac{5}{8}]$.

3. (16 points) (a) Represent the function as power series about $x = 0$. Write out the first five nonzero terms, OR express the series in sigma (Σ) notation.

$$f(x) = \frac{x}{8 + x^3}$$

Answer:

$$\begin{aligned}\frac{x}{8+x^3} &= \frac{x}{8} \cdot \frac{1}{1-(-x^3/8)} = \frac{x}{8}\sum_{n=0}^{\infty}\left(\frac{-x^3}{8}\right)^n \\ &= \frac{x}{8}\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n}}{8^n} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{8^{n+1}} \\ &= \frac{x}{8} - \frac{x^4}{8^2} + \frac{x^7}{8^3} - \frac{x^{10}}{8^4} + \frac{x^{13}}{8^5} - \dots .\end{aligned}$$

(b) Find the radius and interval of convergence for the series you found.

Answer:

The series was found by using the power series representation of $1/(1-u)$, with $u = -x^3/8$. This converges only for $|u| < 1$, and diverges otherwise. So our power series converges if

$$\begin{aligned}|-x^3/8| &< 1 \\ |x^3| &< 8 \\ |x| &< 2.\end{aligned}$$

So the radius of convergence is $R = 2$ and the interval of convergence is $I = (-2, 2)$.

4. (15 points) Represent the integral as a power series and find the radius of convergence. Write out the first five nonzero terms, OR express the series in sigma (Σ) notation.

$$\int x \arctan(8x^3)\, dx$$

Answer:

$$\begin{aligned}
\int x \arctan(8x^3)\, dx &= \int \left[x \sum_{n=0}^{\infty} (-1)^n \frac{(8x^3)^{2n+1}}{2n+1} \right] dx \\
&= \int \left[x \sum_{n=0} (-1)^n \frac{8^{2n+1} x^{6n+3}}{2n+1} \right] dx \\
&= \int \left[\sum_{n=0}^{\infty} (-1)^n \frac{8^{2n+1} x^{6n+4}}{2n+1} \right] dx \\
&= \sum_{n=0}^{\infty} \left[(-1)^n \frac{8^{2n+1}}{2n+1} \int x^{6n+4}\, dx \right] \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{8^{2n+1} x^{6n+5}}{(2n+1)(6n+5)} \\
&= C + \frac{8x^5}{5} - \frac{8^3 x^{11}}{3 \cdot 11} + \frac{8^5 x^{17}}{5 \cdot 17} - \frac{8^7 x^{23}}{7 \cdot 23} + \frac{8^9 x^{29}}{9 \cdot 29} - \dots
\end{aligned}$$

The series was found by using the power series representation of $\arctan(u)$, with $u = 8x^3$. This series converges for $|u| < 1$, and diverges if $|u| > 1$, so our power series converges if $|8x^3| < 1$. This is equivalent to $|x| < \frac{1}{2}$. So the radius of convergence is $R = \frac{1}{2}$.

5. (18 points) (a) Find the Taylor series expansion of $f(x) = \sin(x)$ around $a = \pi/2$. Write your answer in sigma (Σ) notation.

Answer:

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin(x)$	+1
1	$\cos(x)$	0
2	$-\sin(x)$	-1
3	$-\cos(x)$	0
4	$\sin(x)$	+1

$\vdots$

So the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(x-a)^n = 1 + 0 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}.$$

(b) Find the radius and interval of convergence of the series you found.

Answer:

We MAY NOT assume the radius of convergence is unchanged from the Maclaurin series of $\sin(x)$ or $\cos(x)$. Instead we use the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-\pi/2)^{2(n+1)}/(2(n+1))!}{(-1)^n (x-\pi/2)^{2n}/(2n)!} \right|$$
$$= \lim_{n\to\infty} \frac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0.$$

Since $0 < 1$ for all x, the radius of convergence is $R = \infty$. This means the interval of convergence is $(-\infty, \infty)$.

6. (12 points) (a) Use Maclaurin series to evaluate the limit

$$\lim_{x\to 0} \frac{4x\ln(1+x^3) - 4x^4}{x^7}.$$

Answer:

$$\begin{aligned}
\lim_{x\to 0} \frac{4x\ln(1+x^3) - 4x^4}{x^7} &= \lim_{x\to 0} \frac{4x}{x^7}(\ln(1+x^3) - x^3) \\
&= \lim_{x\to 0} \frac{4}{x^6}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x^3)^n}{n} - x^3\right] \\
&= \lim_{x\to 0} \frac{4}{x^6}\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(x^3)^n}{n} \\
&= \lim_{x\to 0} \sum_{n=2}^{\infty}(-1)^{n-1}\frac{4x^{3n-6}}{n} \\
&= \lim_{x\to 0} -2 + \frac{4x^3}{3} - \frac{4x^6}{4} + \frac{4x^9}{5} - \dots \\
&= -2.
\end{aligned}$$

(b) Find the exact value of the sum

$$1 - \ln(2) + \frac{(\ln(2))^2}{2!} - \frac{(\ln(2))^3}{3!} + \dots .$$

Answer:

The sum can be written as

$$\sum_{n=0}^{\infty}(-1)^n\frac{(\ln(2))^n}{n!} = \sum_{n=0}^{\infty}\frac{(-\ln(2))^n}{n!} = e^{-\ln(2)} = 2^{-1} = \frac{1}{2}.$$

Common Maclaurin Series

Function	Series	Initial Terms	Rad./Int. of Convergence
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \dots$	$R = 1, \quad I = (-1, 1)$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$R = \infty, \quad I = (-\infty, \infty)$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$R = \infty, \quad I = (-\infty, \infty)$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6!} + \dots$	$R = \infty, \quad I = (-\infty, \infty)$
$\arctan(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$R = 1, \quad I = [-1, 1]$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$R = 1, \quad I = (-1, 1]$
$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$	$R = 1$