Math 143

Midterm 1 Solutions

March 1, 2018

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Circle your instructor's name:

Yesim Demiroglu George Grell

- **No calculators, notes, or other aids are allowed during this exam.**
- **Show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **You are responsible for checking this exam has all 8 pages.**
- **If possible evaluate trigonometric and logarithmic expressions. Otherwise you do not need to simplify.**

Please copy and sign the following statement.

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

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Signature: ______________________________

1. (15 points) Determine whether the following sequences converge or diverge. If a sequence converges, find its limit. **Justify and show all your work.**

(a)

$$a_n = \frac{2n^2}{\sqrt{3n^4 + n}}$$

Answer:

$$\lim_{n\to\infty} \frac{2n^2}{\sqrt{3n^4+n}} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{2}{\sqrt{3 + 1/n^3}} = \frac{2}{\sqrt{3}}$$

So the sequence converges to $\frac{2}{\sqrt{3}}$.

(b)

$$a_n = \sin(e^{-3n})$$

Answer:

$$\lim_{n\to\infty} \sin(e^{-3n}) = \sin\left(\lim_{n\to\infty} e^{-3n}\right) = \sin(0) = 0$$

So the sequence converges to 0.

(c)

$$a_n = \ln(4n^3 - 3) - \ln(n+1)$$

Answer:

$$\lim_{n\to\infty} \ln(4n^3 - 3) - \ln(n+1) = \lim_{n\to\infty} \ln\left(\frac{4n^3-3}{n+1}\right)$$

Since $\lim_{n\to\infty} \frac{4n^3-3}{n+1} = \infty$ and $\lim_{x\to\infty} \ln(x) = \infty$, this sequence diverges.

2. (10 points) Determine whether the following series converge or diverge. If a series converges, find its sum. **Justify and show all your work.**

(a)

$$\sum_{n=1}^{\infty}\left(\frac{2}{3^n}+\frac{1}{2^n}\right)$$

Answer:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{2}{3^n}+\frac{1}{2^n}\right) &= \sum_{n=1}^{\infty}\frac{2}{3^n}+\sum_{n=1}^{\infty}\frac{1}{2^n}\\ &= \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{3^{n-1}}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\\ &= \frac{2}{3}\cdot\frac{1}{1-1/3}+\frac{1}{2}\cdot\frac{1}{1-1/2}\\ &= 1+1=2\end{aligned}$$

So the series converges to 2.

(b)

$$\sum_{n=1}^{\infty}\frac{n^2}{n^2+4n+4}$$

Answer:

Since $\lim_{n\to\infty}\frac{n^2}{n^2+4n+4}=\lim_{n\to\infty}\frac{1}{1+4/n+4/n^2}=1\neq 0$, the series diverges by the divergence test.

3. (12 points) Use the integral test to determine whether the series converges or diverges. You may assume the terms are decreasing. **Justify and show all your work.**

$$\sum_{n=4}^{\infty} \frac{1}{n\ln(n)(\ln(\ln(n)))}$$

Answer:

Since x, $\ln(x)$, and $\ln(\ln(x))$ are positive, continuous, and non-zero for $x \geq 4$, the function

$$f(x) = \frac{1}{x\ln(x)\ln(\ln(x))}$$

is also positive and continuous for $x \geq 4$. This means the integral test may be applied. We have

$$\begin{aligned}
\int_4^{\infty} f(x)\,dx &= \lim_{t\to\infty} \int_4^t \frac{1}{x\ln(x)\ln(\ln(x))}\,dx \\
&= \lim_{t\to\infty} \int_{\ln(\ln(4))}^{\ln(\ln(t))} \frac{1}{u}\,du \qquad (\text{let } u = \ln(\ln(x))) \\
&= \lim_{t\to\infty} \ln(u)\Big|_{\ln(\ln(4))}^{\ln(\ln(t))} \\
&= \lim_{t\to\infty} \ln(\ln(\ln(t))) - \ln(\ln(\ln(4))) \\
&= \infty.
\end{aligned}$$

Since the integral diverges, the integral test says that the series must also diverge.

4. (10 points) Use the **comparison test** or the **limit comparison test** to determine whether the following series converge or diverge. **Justify and show all your work.**

(a)

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

Answer:

For $n \geq 2$, notice that $n^n \geq 2^n$. Thus $\frac{1}{n^n} \leq \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series ($r = \frac{1}{2}$), the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is also convergent by the comparison test.

Note: A comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is equally straightforward.

(b)

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{3^n - 2}$$

Answer:

We will use the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n}$ (a divergent geometric series with $r = \frac{5}{3}$).

$$\begin{aligned}
\lim_{n\to\infty} \frac{5^n/3^n}{5^{n+1}/3^n - 2} &= \lim_{n\to\infty} \frac{5^n}{5^{n+1}} \cdot \frac{3^n - 2}{3^n} \\
&= \lim_{n\to\infty} \frac{1}{5} \left(\frac{3^n}{3^n} - \frac{2}{3^n} \right) \\
&= \frac{1}{5} \lim_{n\to\infty} 1 - \frac{2}{3^n} \\
&= \frac{1}{5}.
\end{aligned}$$

Since $0 < \frac{1}{5} < \infty$, we may apply the limit comparison test, and since $\sum_{n=1}^{\infty} \frac{5^n}{3^n}$ diverges, so must our original series.

5. (8 points) Match the series with the appropriate letter (A, B, C, D) so that the limit comparison test may be applied. Then determine whether the series converge or diverge.

$$A.\sum_{n=1}^{\infty}\frac{1}{n}, \qquad B.\sum_{n=1}^{\infty}\frac{1}{n^2}, \qquad C.\sum_{n=1}^{\infty}\frac{1}{n^3}, \qquad D.\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}.$$

That is to say, for each of the following, **circle** A, B, C, or D and **circle** converges or diverges. **You do NOT have to show your work.**

1. $$\sum_{n=1}^{\infty}\frac{n}{n^2+n+1}$$

   [A] B C D

   Converges [Diverges]

2. $$\sum_{n=1}^{\infty}\frac{1}{n^2+3}$$

   A [B] C D

   [Converges] Diverges

3. $$\sum_{n=1}^{\infty}\frac{1}{n-\sqrt{n}}$$

   [A] B C D

   Converges [Diverges]

4. $$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n-1}$$

   A B C [D]

   Converges [Diverges]

6. (15 points) (a) Test the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-n}$$

for convergence or divergence. **Justify and show all your work.**

Answer:

Let $f(x) = xe^{-x}$. Then $f'(x) = e^{-x}(1-x)$. This is negative for $x > 1$, so $f(x)$ is decreasing for $x > 1$. Also

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

by L'Hospital's Rule. So the sequence ne^{-n} is a positive, decreasing sequence that approaches 0. Thus, by the alternating series test, the series $\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-n}$ is a convergent series.

(b) Approximate the series using the first three terms. **DO NOT attempt to convert to a decimal number.**

Answer:

The series is approximately

$$\sum_{n=1}^{3}(-1)^{n+1}ne^{-n} = \frac{1}{e} - \frac{2}{e^2} + \frac{3}{e^3}.$$

(c) Use the Alternating Series Estimation Theorem to estimate the error in this approximation. **DO NOT attempt to convert to a decimal number.**

Answer:

The Alternating Series Estimation Theorem says the error in our estimation is at most

$$|a_4| = \frac{4}{e^4}.$$

7. (10 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. **Justify and show all your work.**

(a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$$

Answer:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{e^n} \right| = \sum_{n=1}^{\infty} \frac{1}{e^n}.$$

This is a convergent geometric series ($r = 1/e$) so the original series is absolutely convergent.

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{3n+5}}$$

Answer:

Let $b_n = \frac{1}{\sqrt{3n+5}}$. Notice that b_n is a positive, decreasing sequence that approaches 0. So

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{3n+5}} = \sum_{n=1}^{\infty} (-1)^n b_n$$

is convergent by the alternating series test. On the other hand,

$$\lim_{n\to\infty} \frac{1/\sqrt{n}}{1/\sqrt{3n+5}} = \lim_{n\to\infty} \frac{\sqrt{3n+5}}{\sqrt{n}} = \sqrt{3}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = 1/2$), the limit comparison test says that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{3n+5}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{3n+5}}$$

is divergent. So the series is not absolutely convergent. A series that is convergent, but not absolutely convergent is conditionally convergent.