Math 143

Final

May 6, 2018

NAME (please print legibly):  

Your University ID Number:  

Circle your instructor’s name:

Yesim Demiroglu    George Grell

• No calculators, notes, or other aids are allowed during this exam.

• Show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.

• You are responsible for checking this exam has all 17 pages.

• If possible evaluate trigonometric and logarithmic expressions. Otherwise you do not need to simplify.

Please copy and sign the following statement.

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

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Part A

1. (20 points) If a sequence below converges, find its limit, and justify by citing any theorems/rules you use. If a sequence below diverges, state whether it diverges because it oscillates, diverges to $+\infty$, or diverges to $-\infty$.

(a) $a_n = \frac{\ln n}{n}$

Let $f(x) = \frac{\ln x}{x}$. Then $\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$, by L'Hopital's rule.

Hence $\lim_{n \to \infty} a_n = 0$, as $a_n = f(n)$.

(b) $a_n = (-1)^n n^3$

As $\lim_{n \to \infty} n^3 = \infty$, $\sum_{n=1}^{\infty} (-1)^n n^3$ diverges because it oscillates.
(c) $a_n = \frac{\cos n}{n}$

Since $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ and $\lim_{n \to \infty} -\frac{1}{n} = \lim_{n \to \infty} \frac{1}{n} = 0$,

$\lim_{n \to \infty} \frac{\cos n}{n} = 0$ by the squeeze theorem.

(d) $a_n = \ln (3n^2 + 11) - \ln (5n^2 + 8n - 2)$

\[
a_n = \ln \left( \frac{3n^2 + 11}{5n^2 + 8n - 2} \right). \quad \text{Since} \quad \lim_{n \to \infty} \frac{3n^2 + 11}{5n^2 + 8n - 2} = \frac{3}{5},
\]

\[
\lim_{n \to \infty} \ln \left( \frac{3n^2 + 11}{5n^2 + 8n - 2} \right) = \ln \left( \frac{3}{5} \right).
\]
2. (14 points) Determine whether the following series converge or diverge. If a series converges, find its sum. Justify and show all your work.

(a) \[ \sum_{n=1}^{\infty} \left( \frac{4}{5^n} + \frac{1}{3^n} \right) \]

Since \[ \sum_{n=1}^{\infty} \frac{4}{5^n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{3^n} \text{ are convergent geometric series}, \]
its sum is \[ \frac{\frac{4}{5}}{1 - \frac{1}{5}} = 1. \] Since \[ \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \cdots, \] its sum is \[ \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{1}{2}. \] Hence \[ \sum_{n=1}^{\infty} \frac{4}{5^n} + \frac{1}{3^n} = 1 + \frac{1}{2}. \]

(b) \[ \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \]

Since \[ \frac{2}{n^2 - 1} = \frac{1}{n-1} - \frac{1}{n+1}, \] this series telescopes.

\[ S_n = (1 - \frac{1}{3}) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \cdots + \left( \frac{1}{n-2} - \frac{1}{n} \right) \]

\[ + \left( \frac{1}{n-1} - \frac{1}{n+1} \right) = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}. \]

Since \[ \lim_{n \to \infty} S_n = 1 + \frac{1}{2}, \] the series converges, and its sum is \[ \frac{3}{2}. \]
\[ \sum_{n=1}^{\infty} \frac{(4)^n}{(3^n)} \]

is a geometric series with common ratio \( r = \frac{3}{4} < 1 \); it always converges, and since \( \lim_{n \to \infty} \frac{4^n}{3^n} = 4 \), we have that \( \frac{n+5}{4^n} \to 0 \).

Further, note that these series converge, and since \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \), we see that it is decreasing, note that \( \frac{e^x}{x^2} \). 

Let \( f(x) = \frac{e^x}{x^2} \).

See that \( \frac{e^x}{x^2} = \frac{x e^x}{x^2} = \frac{e^x}{x} = e^x \). 

Clearly \( f(x) \) is positive on \([1, \infty)\). 

To tests you use, and justifying their use completely:

3. (15 points) Determine whether the following series converge or diverge, naming any tests you use, and justifying their use completely.
(c) \[ \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \]

Let \( f(x) = \frac{1}{x \ln(x)} \). For \( x \geq 2 \), \( \frac{1}{x \ln(x)} > 0 \).

\[ f'(x) = \frac{-(1+\ln(x))}{(x \ln(x))^2} \leq 0 \quad \text{so} \quad f \text{ is decreasing.} \]

\[ \int_{2}^{\infty} \frac{1}{x \ln(x)} \, dx = \lim_{t \to \infty} \ln(\ln(x)) \bigg|_{2}^{t} = \lim_{t \to \infty} \ln(\ln(t)) - \ln(\ln(2)) = \infty. \]

(Using \( u = \ln(x) \))

Hence the integral diverges, so the series also diverges by the integral test.
4. (14 points) Find a power series expansion of the function

\[ f(x) = x \ln(1 - 5x^2). \]

Write the first five non-zero terms, or express in sigma (\(\Sigma\)) notation. What is the radius of convergence?

\[
\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n
\]

\[
x \ln(1 - 5x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-5x^2)^n}{n} (x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 5^n x^{2n+1}}{n}
\]

\[= \sum_{n=1}^{\infty} \frac{-5^n x^{2n+1}}{n} \quad \text{as} \quad (-1)^{2n-1} = -1 \]

Using the root test

\[
\lim_{n \to \infty} \left| \frac{-5^n x^{2n+1}}{n} \right|^{1/n} = |5x^2|.\text{ Set } |5x^2| < 1, \text{ we get } |x| < \frac{1}{\sqrt{5}}.
\]

\[\text{So the radius of convergence is } R = \frac{1}{\sqrt{5}}\]

\[
|X| < \frac{1}{\sqrt{5}}. \text{ So the radius of convergence is } R = \frac{1}{\sqrt{5}}.
\]

When \( x = \frac{1}{\sqrt{5}} \),

\[
\sum_{n=1}^{\infty} \frac{-5^n x^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{\sqrt{5} n}, \text{ which diverges, because it is equal to } \frac{-1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{n} \text{, and } \sum_{n=1}^{\infty} \frac{1}{n}
\]

is the harmonic series. Similarly, the series diverges.

When \( x = -\frac{1}{\sqrt{5}} \), in neither case do we get an alternating series. So the IOC is \((-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})\).

The interval of convergence is not asked for, but here it is.
5. (15 points) Find the Taylor series for \( f(x) = \frac{1}{x} \) around \( a = 4 \) using the definition of a Taylor series. What is the radius of convergence?

\[
f(x) = \frac{1}{x}
\]

\[
f'(x) = -\frac{1}{x^2} \quad f'(4) = -\frac{1}{16}
\]

\[
f''(x) = 2x^{-3} \quad f''(4) = \frac{2}{4^3}
\]

\[
f'''(x) = -6x^{-4} \quad f'''(4) = -\frac{6}{4^4}
\]

\[\vdots\]

\[
f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} (x-4)^n
\]

Using the ratio test:

\[
\lim_{n \to \infty} \left| \frac{(x-4)^{n+1}}{4^{n+2}} \cdot \frac{4^{n+1}}{(x-4)^n} \right| = \lim_{n \to \infty} \left| \frac{x-4}{4} \right| < 1
\]

\[|x-4| < 4, \text{ so } R = 4.
\]
6. (12 points) Find the radius and interval of convergence of the power series below.

(a) \[ \lim_{n \to \infty} \left| \frac{7^n (x-4)^{2n+1}}{(n+1)(2n+3)!} \right| = \lim_{n \to \infty} \left| \frac{7^n (x-4)^{2n+1}}{7^n(x-4)^{2n+1}} \right| = \lim_{n \to \infty} \left| \frac{7(x-4)^2}{(2n+1)(2n+2)} \right| = 0. \] So \( R = \infty \), \( \text{IOC} = (-\infty, \infty) \).

(b) \[ \lim_{n \to \infty} \left| \frac{(n+1)!}{5^n \sqrt{n+4}} \right| = \lim_{n \to \infty} \sqrt{\frac{n+1}{5^n}} \left( \frac{3x+2}{n!(3x+2)^n} \right) \left( \sqrt{n+3} \right) = \lim_{n \to \infty} \sqrt{n+3} \left( \frac{n+1}{5^n} \right) |3x+2| = \infty. \] So \( R = 0 \), \( \text{IOC} = \frac{5-2}{3} \).
Part B

7. (15 points) Consider the parametric curve defined by \( x = t^4 + 1, y = t^3 - t. \)

(a) Find \( dy/dx. \) For what values of \( t \) is the tangent line vertical or horizontal?

\[
\frac{dy}{dt} = 3t^2 - 1 \quad \rightarrow \quad \frac{dy}{dx} = \frac{3t^2 - 1}{4t^3}
\]

\[
\frac{dx}{dt} = 4t^3
\]

\[
\frac{dy}{dx} = 0 \quad \text{if} \quad t = \pm \sqrt[3]{\frac{1}{3}} \quad \text{so the tangent line is horizontal}
\]

The tangent line is vertical when \( t = 0. \)

(b) At the point \((2, 0)\) the curve has two different tangent lines. Find both tangent lines at that point.

\[
x = 2, \quad y = 0 \quad \text{at} \quad t = 1 \quad \text{and} \quad t = -1
\]

If \( t = 1, \) \( \frac{dy}{dx} = \frac{1}{2}, \) so the line is \( y = \frac{1}{2}(x - 2) \)

If \( t = -1, \) \( \frac{dy}{dx} = -\frac{1}{2}, \) so the line is \( y = -\frac{1}{2}(x - 2) \)
(c) Determine the intervals (in $t$) where the curve is concave up, and where it is concave down.

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = -\frac{6t^4 + 12t^2}{64t^7} = \frac{6 - 3t^2}{32t^5}
\]

Changes in concavity occur at $t = 0, t = \pm \sqrt{2}$.

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<tr>
<td>$\sqrt{2}$</td>
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Concave up: $(-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$

Concave down: $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$
8. (12 points) Consider the parametric curve defined by \( x = t - 2\sin(t), y = 1 - 2\cos(t) \) on the interval \( \pi/2 \leq t \leq 3\pi/2 \).

(a) Find the area underneath the curve on the given interval.

Since \( \frac{dx}{dt} = 1 - 2\cos t \) and \( \cos t \leq 0 \) in quadrants 2 and 3, \( \frac{dx}{dt} < 0 \). Hence the curve moves from left to right and so

\[
A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - 2\cos t)(1 - 2\cos t) \, dt
\]

\[
= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - 4\cos t + 4\cos^2 t) \, dt
\]

\[
= \left[ t - 4\sin t + \frac{4}{2} \left( \frac{t}{2} + \frac{\sin^2 t}{4} \right) \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}
\]

\[
= \frac{3\pi}{2} - 4 - \left( \frac{3\pi}{2} - 4 \right) = 8 + \frac{6\pi}{2} = 8 + 3\pi
\]

(b) Set up, but DO NOT EVALUATE an integral that gives the arc length of the curve on the given interval.

We know the curve does not cross itself, because \( \frac{dy}{dt} > 0 \).

\[
\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{(1 - 2\cos t)^2 + (2\sin t)^2} \, dt
\]
9. (16 points)

(a) Convert the polar coordinates \((-3, \pi/3)\) and \((2, -\pi/6)\) to Cartesian coordinates.

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

\[
\left( r, \theta \right) = \left( -3, \frac{\pi}{3} \right)
\]

corresponds to

\[
\left( -3 \left( \frac{1}{2} \right), -3 \left( \frac{\sqrt{3}}{2} \right) \right) = (x, y)
\]

\[
\left( r, \theta \right) = \left( 2, -\frac{\pi}{6} \right)
\]

\[
\left( 2 \left( \frac{\sqrt{3}}{2} \right), 2 \left( -\frac{1}{2} \right) \right) = (\sqrt{3}, -1) = (x, y)
\]

(b) Convert the Cartesian coordinates \((1, \sqrt{3})\) and \((-2, 2)\) to polar coordinates. Make sure 
\(r \geq 0\) and \(0 \leq \theta < 2\pi\).

\[
(1, \sqrt{3})
\]

\[
\sqrt{3} = \frac{\sqrt{3}}{x} = \tan \theta
\]

so \(\theta = \frac{\pi}{3}\)

\[
x^2 + y^2 = 1 + 3 = 4 = r^2
\]

polar point:

\[
\left( \sqrt{3}, \frac{\pi}{3} \right)
\]

\[
(-2, 2)
\]

\[
tan \theta = -1, \quad \text{so} \quad \theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}
\]

since \(x < 0, y > 0\), we read

\(\theta = \frac{3\pi}{4}\).

\[
x^2 + y^2 = 8, \quad \text{so} \quad r = 2\sqrt{2}
\]

\[
(2\sqrt{2}, \frac{3\pi}{4})
\]
10. (12 points) Consider the polar curve $r = 1 + 2 \cos(\theta)$

(a) Sketch the curve on the provided graph.

(b) For what values of $\theta$ does the curve cross itself?

\begin{align*}
\text{(b) The curve crosses itself when } & 1 + 2 \cos(\theta) = 0, \text{ or} \\
\theta &= \frac{2\pi}{3}, \frac{5\pi}{3} \text{ or } \end{align*}
11. (15 points) Find the area inside the polar curve \( r = 3 + \cos 4\theta \). A plot is given below for reference.

\[
\begin{align*}
A &= 8 \int_0^{\pi/4} \frac{(3 + \cos 4\theta)^2}{2} \, d\theta \\
&= 4 \int_0^{\pi/4} 9 + 6 \cos 4\theta + \cos^2(4\theta) \, d\theta \\
&= 4 \int_0^{\pi/4} 9 + 6 \cos 4\theta + \frac{1 + \cos 8\theta}{2} \, d\theta \\
&= 4 \int_0^{\pi/4} 9.5 + 6 \cos 4\theta + \frac{\cos 8\theta}{2} \, d\theta \\
&= 4 \left[ 9.5\theta + \frac{6\sin 4\theta}{4} + \frac{\sin 8\theta}{16} \right]_0^{\pi/4} \\
&= (9.5)(\pi/4)(4) = 9.5\pi
\end{align*}
\]