Math 143

Final

May 6, 2018

NAME (please print legibly): Key

Your University ID Number: ______

Circle your instructor's name:

Yesim Demiroglu   George Grell

- **No calculators, notes, or other aids are allowed during this exam.**
- **Show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **You are responsible for checking this exam has all 17 pages.**
- **If possible evaluate trigonometric and logarithmic expressions. Otherwise you do not need to simplify.**

Please copy and sign the following statement.

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

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Signature: ______

Part A		
QUESTION	VALUE	SCORE
1	20	
2	14	
3	15	
4	14	
5	15	
6	12	
TOTAL	90	

Part B		
QUESTION	VALUE	SCORE
7	15	
8	12	
9	16	
10	12	
11	15	
TOTAL	70	

Part A

1. (20 points) If a sequence below converges, find its limit, and justify by citing any theorems/rules you use. If a sequence below diverges, state whether it diverges because it oscillates, diverges to $+\infty$, or diverges to $-\infty$.

(a) $a_n = \frac{\ln n}{n}$

Let $f(x) = \frac{\ln x}{x}$. Then $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$, by L'Hopital's rule.

Hence $\lim_{n\to\infty} a_n = 0$, as $a_n = f(n)$.

(b) $a_n = (-1)^n n^3$

As $\lim_{n\to\infty} n^3 = \infty$, $\{(-1)^n n^3\}_{n=1}^{\infty}$ diverges because it oscillates.

(c) $a_n = \frac{\cos n}{n}$

Since $\frac{-1}{n} \leq \frac{\cos n}{n} < \frac{1}{n}$ and $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$,

$\lim_{n\to\infty} \frac{\cos n}{n} = 0$ by the squeeze theorem.

(d) $a_n = \ln(3n^2 + 11) - \ln(5n^2 + 8n - 2)$

$a_n = \ln\left(\frac{3n^2+11}{5n^2+8n-2}\right)$. Since $\lim_{n\to\infty}\left(\frac{3n^2+11}{5n^2+8n-2}\right) = \frac{3}{5}$,

$\lim_{n\to\infty} \ln\left(\frac{3n^2+11}{5n^2+8n-2}\right) = \ln\left(\frac{3}{5}\right)$.

2. (14 points) Determine whether the following series converge or diverge. If a series converges, find its sum. Justify and show all your work.

(a)

$$\sum_{n=1}^{\infty}\left(\frac{4}{5^n}+\frac{1}{3^n}\right)$$

Since $\sum_{n=1}^{\infty} \frac{4}{5^n}$ and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ are convergent geometric series, $\sum_{n=1}^{\infty} \frac{4}{5^n}+\frac{1}{3^n}$ converges. Since $\sum_{n=1}^{\infty} \frac{4}{5^n} = \frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\dots$, its sum is $\frac{\frac{4}{5}}{1-\frac{1}{5}} = 1$. Since $\sum_{n=1}^{\infty}\frac{1}{3^n} = \frac{1}{3}+\frac{1}{9}+\dots$, its sum is $\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$. Hence $\sum_{n=1}^{\infty} \frac{4}{5^n}+\frac{1}{3^n} = 1+\frac{1}{2}$.

(b)

$$\sum_{n=2}^{\infty}\frac{2}{n^2-1}$$

Since $\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$, this series telescopes.

$$S_n = \left(1-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\dots+\left(\cancel{\frac{1}{n-2}}-\frac{1}{n}\right)$$

$$+\left(\cancel{\frac{1}{n-1}}-\frac{1}{n+1}\right) = 1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}.$$

Since $\lim_{n\to\infty} S_n = 1+\frac{1}{2}$, the series converges, and its sum is $\frac{3}{2}$.

3. (15 points) Determine whether the following series converge or diverge, naming any tests you use, and justifying their use completely.

(a)

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

Let $f(x) = \frac{e^{\frac{1}{x}}}{x^2}$. Clearly f is positive on $[1, \infty)$. To see that it is decreasing, note that

$$f'(x) = \frac{x^2 e^{\frac{1}{x}}\left(\frac{-1}{x^2}\right) - e^{\frac{1}{x}}(2x)}{x^4} = \frac{e^{\frac{1}{x}}(-1-2x)}{x^4} < 0.$$

$$\lim_{t\to\infty} \int_1^t \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty} -e^{\frac{1}{x}} \Big|_1^t = \lim_{t\to\infty} -e^{\frac{1}{t}} - (-e^1) = e-1.$$

(using $u = \frac{1}{x}$)

Since $e-1$ is finite, $\int_1^\infty f(x)dx$ converges. Hence $\sum f(n)$ converges as well by the integral test.

(b)

$$\sum_{n=1}^{\infty} \frac{n4^n + 5}{(3.9)^n - n}$$

Since $n4^n + 5 > 4^n$ and $(3.9)^n - n < 3.9^n$, we have that $\frac{n4^n+5}{3.9^n - n} > \left(\frac{4}{3.9}\right)^n$. Further both the given series and $\sum_{n=1}^{\infty}\left(\frac{4}{3.9}\right)^n$ have positive terms. Since $\sum_{n=1}^{\infty}\left(\frac{4}{3.9}\right)^n$ is a geometric series with common ratio $r = \frac{4}{3.9} > 1$, it diverges, and $\sum_{n=1}^{\infty} \frac{n4^n+5}{3.9^n - n}$ diverges by comparison.

(c)

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

Let $f(x) = \frac{1}{x \ln x}$. For $x \geq 2$, $\frac{1}{x \ln x} > 0$.

$f'(x) = \frac{-(1+\ln x)}{(x \ln x)^2} < 0$, so f is decreasing.

$\int_2^\infty \frac{1}{x \ln x} = \lim_{t\to\infty} \ln(\ln(x)) \Big|_2^t = \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln 2) = \infty.$

(using $u = \ln x$)

Hence the integral diverges, so the series also diverges by the integral test.

4. (14 points) Find a power series expansion of the function

$$f(x) = x\ln(1-5x^2).$$

Write the first five non-zero terms, or express in sigma (Σ) notation. What is the radius of convergence?

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$x \ln(1-5x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-5x^2)^n (x)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1+n} 5^n x^{2n+1}}{n}$$

$$= \sum_{n=1}^{\infty} \frac{-5^n x^{2n+1}}{n} \qquad \left(\text{as } (-1)^{2n-1} = -1\right)$$

Using the root test

$\lim_{n\to\infty} \left| \frac{-5^n x^{2n+1}}{n} \right|^{\frac{1}{n}} = |5x^2|$. Set $|5x^2|<1$, we get $|x| < \frac{1}{\sqrt{5}}$. So the radius of convergence is $\underline{\underline{R = \frac{1}{\sqrt{5}}}}$

When $x = \frac{1}{\sqrt{5}}$, $\sum_{n=1}^{\infty} \frac{-5^n x^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{\sqrt{5}\,n}$, which diverges, because it is equal to $\frac{-1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{n}$, and $\Sigma \frac{1}{n}$ is the harmonic series. Similarly, the series diverges when $x = \frac{-1}{\sqrt{5}}$ — in neither case do we get an alternating series. So the IOC is $\left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

The interval of convergence is not asked for, but here it is.

5. (15 points) Find the Taylor series for $f(x) = \dfrac{1}{x}$ around $a = 4$ using the definition of a Taylor series. What is the radius of convergence?

$f(4) = \frac{1}{4}$

$f'(x) = \frac{-1}{x^2}$ $\quad f'(4) = \frac{-1}{16}$

$f''(x) = 2x^{-3}$ $\quad f''(4) = \frac{2}{4^3}$

$f'''(x) = -6x^{-4}$ $\quad f'''(4) = \frac{-6}{4^4}$

$\vdots$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!\, 4^{n+1}} (x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} (x-4)^n$$

Using the ratio test:

$$\lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{4^{n+2}} \cdot \frac{4^{n+1}}{(x-4)^n} \right| = \lim_{n\to\infty} \frac{|x-4|}{4} < 1$$

$|x-4| < 4$, so $R = 4$.

6. (12 points) Find the radius and interval of convergence of the power series below.

(a)

$$\sum_{n=1}^{\infty} \frac{7^n(x-4)^{2n+1}}{n(2n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{7^{n+1}(x-4)^{2n+3}}{(n+1)(2n+3)!} \cdot \frac{n(2n+1)!}{7^n(x-4)^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{7(x-4)^2}{(2n+3)(2n+2)} \left(\frac{n}{n+1}\right) \right|$$

$= 0$. So $R = \infty$, $IOC = (-\infty, \infty)$.

(b)

$$\sum_{n=1}^{\infty} \frac{n!(-1)^n(3x+2)^n}{5^n\sqrt{n+3}}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)!(3x+2)^{n+1}}{5^{n+1}\sqrt{n+4}} \cdot \frac{5^n\sqrt{n+3}}{n!(3x+2)^n} \right| = \lim_{n\to\infty} \sqrt{\frac{n+3}{n+4}} \frac{(n+1)}{5} |3x+2| = \infty.$$

So $R = 0$, and $IOC = \left\{\frac{-2}{3}\right\}$

Part B

7. (15 points) Consider the parametric curve defined by $x = t^4 + 1, y = t^3 - t$.

(a) Find dy/dx. For what values of t is the tangent line vertical or horizontal?

$$\left.\begin{aligned} \frac{dy}{dt} &= 3t^2 - 1 \\ \frac{dx}{dt} &= 4t^3 \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{3t^2-1}{4t^3}$$

$\frac{dy}{dx} = 0$ if $t = \pm \frac{1}{\sqrt{3}}$, so the tan line is horizontal

The tan line is vertical when $t = 0$.

(b) At the point $(2, 0)$ the curve has two different tangent lines. Find both tangent lines at that point.

$x = 2, y = 0$ at $t = 1$ and $t = -1$

If $t=1$, $\frac{dy}{dx} = \frac{1}{2}$, so the line is $y = \frac{1}{2}(x-2)$

If $t=-1$, $\frac{dy}{dx} = \frac{-1}{2}$ so the line is $y = \frac{-1}{2}(x-2)$

(c) Determine the intervals (in t) where the curve is concave up, and where it is concave down.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-6t^4+12t^2}{64t^7} = \frac{6-3t^2}{32t^5}$$

Changes in concavity can occur at $t=0$, $t=\pm\sqrt{2}$

	$-\sqrt{2}$		0		$\sqrt{2}$	
$6-3t^2$	$-$	$+$		$+$		$-$
t^5	$-$	$-$		$+$		$+$
	$+$	$-$		$+$		$-$

concave up: $(-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$

concave down: $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$

8. (12 points) Consider the parametric curve defined by $x = t - 2\sin(t), y = 1 - 2\cos(t)$ on the interval $\pi/2 \le t \le 3\pi/2$.

(a) Find the area underneath the curve on the given interval.

Since $\frac{dx}{dt} = 1 - 2\cos t$ and $\cos t < 0$ in quadrants 2 and 3, $\frac{dx}{dt} > 0$. Hence the curve moves from left to right and so $A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \underbrace{(1-2\cos t)}_{y(t)} \underbrace{(1-2\cos t)}_{\frac{dx}{dt}} dt$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 - 4\cos t + 4\cos^2 t \, dt$$

$$= t - 4\sin t + 4\left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$\star$

$$= \frac{9\pi}{2} + 4 - \left(\frac{3\pi}{2} - 4\right) = 8 + \frac{6\pi}{2} = 8 + 3\pi$$

$\star$ $\cos^2 t = \frac{1 + \cos 2t}{2}$

$\int \cos^2 t \, dt = \frac{t}{2} + \frac{\sin 2t}{4}$

(b) Set up, but DO NOT EVALUATE an integral that gives the arc length of the curve on the given interval.

We know the curve does not cross itself, because $\frac{dx}{dt} > 0$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{(1-2\cos t)^2 + (2\sin t)^2} \, dt$$

9. (16 points)

(a) Convert the polar coordinates $(-3, \pi/3)$ and $(2, -\pi/6)$ to Cartesian coordinates.

$x = r\cos t, \quad y = r\sin t$

$(r,\theta) = (-3, \frac{\pi}{3})$ corresponds to $\left(-3\left(\frac{1}{2}\right), -3\left(\frac{\sqrt{3}}{2}\right)\right) = (x,y)$

$(r,\theta) = (2, \frac{-\pi}{6})$ " " $\left(2\left(\frac{\sqrt{3}}{2}\right), 2\left(\frac{-1}{2}\right)\right) = (\sqrt{3}, -1) = (x,y)$

(b) Convert the Cartesian coordinates $(1, \sqrt{3})$ and $(-2, 2)$ to polar coordinates. Make sure $r \geq 0$ and $0 \leq \theta < 2\pi$.

$(1, \sqrt{3})$

$\sqrt{3} = \frac{y}{x} = \tan\theta$

So $\theta = \frac{\pi}{3}$

$x^2 + y^2 = 1 + 3 = 4 = r^2$

polar point:

$\left(2, \frac{\pi}{3}\right)$

$(-2, 2)$

$\tan\theta = -1$, so $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

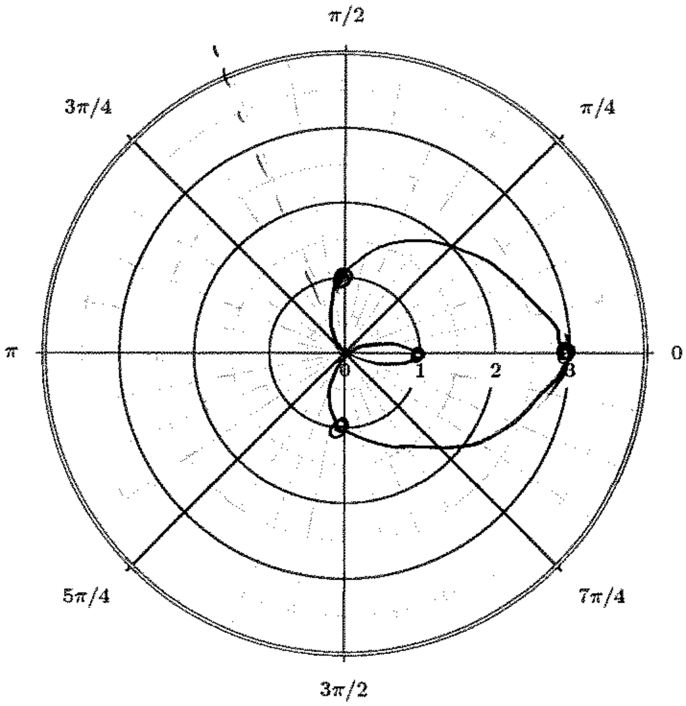
Since $x < 0$, $y > 0$, we need

$\theta = \frac{3\pi}{4}$.

$x^2 + y^2 = 8$, so $r = 2\sqrt{2}$

$\left(2\sqrt{2}, \frac{3\pi}{4}\right)$

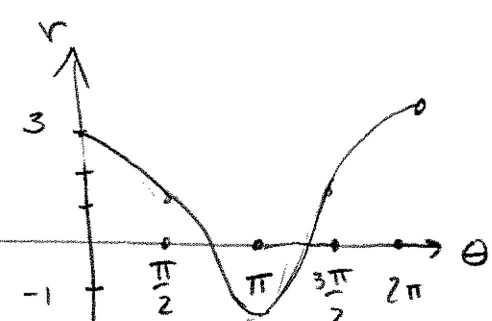
10. (12 points) Consider the polar curve $r = 1 + 2\cos(\theta)$

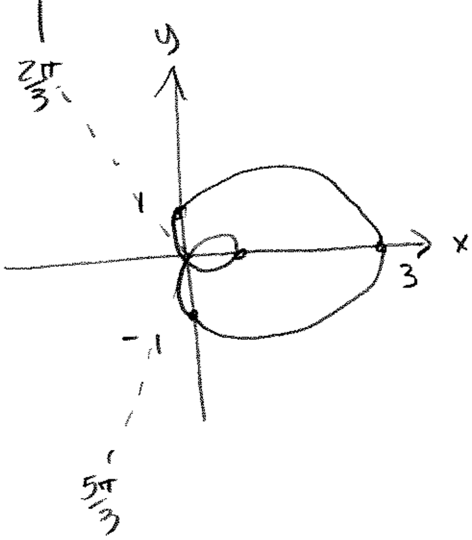


(a) Sketch the curve on the provided graph.

(b) For what values of θ does the curve cross itself?

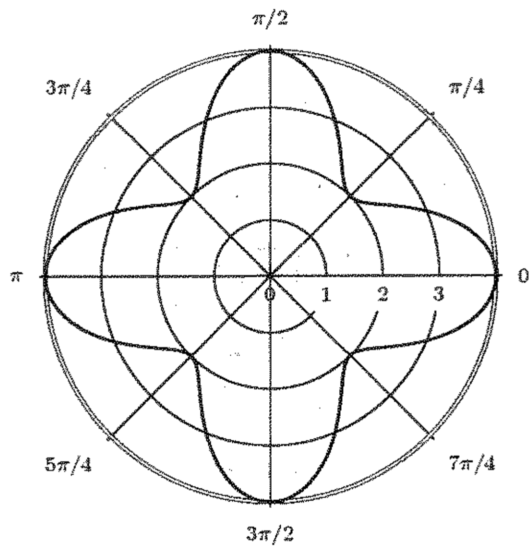
(a)





(b) the curve crosses itself when $1 + 2\cos\theta = 0$, or $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$

11. (15 points) Find the area inside the polar curve $r = 3 + \cos 4\theta$. A plot is given below for reference.



$$A = 8 \int_0^{\frac{\pi}{4}} \frac{(3+\cos 4\theta)^2}{2}\, d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} 9 + 6\cos 4\theta + \cos^2(4\theta)\, d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} 9 + 6\cos 4\theta + \frac{1+\cos 8\theta}{2}\, d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} 9.5 + 6\cos 4\theta + \frac{\cos 8\theta}{2}\, d\theta$$

$$= 4\left(9.5\theta + \frac{6\sin 4\theta}{4} + \frac{\sin 8\theta}{16}\right\Big|_0^{\frac{\pi}{4}} = (9.5)\left(\frac{\pi}{4}\right)(4) = 9.5\pi$$