Math 143

Midterm 2

November 20, 2018

Name: Key

UR ID: ____________________

Circle your Instructor's Name:

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- No calculators, other electronic devices, or notes are permitted during the exam.
- Work should be shown and justification offered for each question on the exam. Credit may not be granted for unjustified answers, even if they are correct.

Please Initial:_______

PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. (15 points)

(a) (10 pts) Use the integral test to show the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}e^{\sqrt{n}}}.$$

We may use the integral test, because $\left\{\frac{1}{\sqrt{n}\, e^{\sqrt{n}}}\right\}$ is a positive decreasing sequence.

Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$, so $2\,du = \frac{1}{\sqrt{x}}dx$.

Then $\lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}\, e^{\sqrt{x}}}dx = \lim_{t\to\infty} \int_{\sqrt{1}}^{\sqrt{t}} \frac{2}{e^u}du = \lim_{t\to\infty} -2e^{-u}\Big|_1^{\sqrt{t}}$

$= \lim_{t\to\infty}\left(-2e^{-\sqrt{t}} - (-2e^{-1})\right) = \frac{2}{e}$. Since $\frac{2}{e}$ is finite,

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}e^{\sqrt{n}}}$ is convergent.

$\boxed{\text{Note: The substitution } u = e^{\sqrt{x}} \text{ also works.}}$

(b) (5 pts) If we use

$$S_4 = \frac{1}{e} + \cdots + \frac{1}{2e^2}$$

to estimate the sum of the series, what is the upper bound on the error **given by the integral test**? (Recall that the error is R_4, which equals $S - S_4$.)

$R_4 \leq \int_4^{\infty} \frac{1}{\sqrt{x}\, e^{\sqrt{x}}}dx = \lim_{t\to\infty} -2e^{-\sqrt{x}}\Big|_4^t = 2e^{-\sqrt{4}} = \frac{2}{e^2}.$

2. (15 points)

Consider the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{1/n}}{n}.$$

(a) (12 pts) **Use the alternating series test** to determine if the series converges. (Credit will not be offered if you use another test.)

This series alternates, as $\frac{e^{\frac{1}{n}}}{n} > 0$ for all n.

$\lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n} < \lim_{n\to\infty} \frac{e}{n} = e\left(\lim_{n\to\infty} \frac{1}{n}\right) = e\cdot 0 = 0.$

Let $f(x) = \frac{e^{\frac{1}{x}}}{x}$. Then $f'(x) = \frac{xe^{\frac{1}{x}}\left(\frac{-1}{x^2}\right) - e^{\frac{1}{x}}}{x^2} = \frac{e^{\frac{1}{x}}\left(-\frac{1}{x} - 1\right)}{x^2} < 0.$

Hence $\left\{\frac{e^{\frac{1}{x}}}{x}\right\}$ is decreasing.

We conclude that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{\frac{1}{n}}}{n}$ converges.

(b) (3 pts) Is the series conditionally convergent, absolutely convergent, or both? Why?

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}e^{\frac{1}{n}}}{n}\right| = \sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n}$. Since $e^{\frac{1}{n}} > 1$, we have

that $\frac{e^{\frac{1}{n}}}{n} > \frac{1}{n} > 0$. Since $\sum \frac{1}{n}$ is divergent (harmonic),

we use the comparison test to conclude that

$\sum \frac{e^{\frac{1}{n}}}{n}$ diverges. Hence $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{\frac{1}{n}}}{n}$ is conditionally

convergent.

3. (50 points) Determine whether or not the following series converge or diverge. **If they converge state if they converge conditionally or absolutely.** Show your work and **state which test you use.**

Here is an example showing what a correct answer should look like:

$$\sum_{n=1}^{\infty} \frac{1}{n^5+1}$$

Answer: The terms of this series satisfy

$$0 < \frac{1}{n^5+1} < \frac{1}{n^5}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^5}$ is a convergent p-series ($p = 5 > 1$), we may conclude that the given series converges by the comparison test. This series is absolutely convergent, because its terms are positive.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 \ln(n+1)}$

$$0 < \frac{1}{n^2 \ln(n+1)} < \frac{1}{n^2} \quad \text{for} \quad n \geq 2.$$

(Note that $\frac{1}{\ln(2)} > 1$, as $\ln(2) < 1$, so the inequality does not hold when $n=1$.)

Since $\sum \frac{1}{n^2}$ is a convergent p-series ($p=2>1$), we have that $\sum \frac{1}{n^2 \ln(n+1)}$ converges by comparison.

Since the series has positive terms, the convergevee is absolute.

(b) $\sum_{n=1}^{\infty} n^{1/n}$

$\lim_{n\to\infty} \sqrt[n]{n} = 1 \neq 0$. By the test for divergence, the series diverges.

(c) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

Using the root test,

$\lim_{n\to\infty} \sqrt[n]{\left|\frac{-2}{n}\right|^n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$. This series converges absolutely.

(d) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$

Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)^2\,2^{n+1}}{(n+1)!}\cdot\frac{n!}{(-1)^{n+1}n^2 2^n}\right|$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\frac{2}{n+1}=0<1.$$

This series converges absolutely.

(e) $\sum_{n=1}^{\infty}\frac{\sqrt{5n-4}}{n+3}$

$\frac{\sqrt{5n-4}}{n+3}>0$ for all n.

Using the limit comparison test,

$$\lim_{n\to\infty}\frac{\frac{\sqrt{5n-4}}{n+3}}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{\sqrt{n}\sqrt{5n-4}}{n+3}=\lim_{n\to\infty}\frac{\cancel{\sqrt{n^2}}\sqrt{5-\frac{4}{n}}}{\cancel{n}(1+\frac{3}{n})}=\sqrt{5}$$

Since $\sqrt{5}$ is finite and non zero, and since

$\sum\frac{1}{\sqrt{n}}$ is a divergent p-series, $\left(p=\frac{1}{2}<1\right)$,

$\sum_{n=1}^{\infty}\frac{\sqrt{5n-4}}{n+3}$ diverges as well.

4. (20 points) Find the interval of convergence of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$ Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n x^n}{n^2}\right|} = \lim_{n\to\infty} \frac{2|x|}{\sqrt[n]{n^2}} = 2|x|.$$

Setting $2|x| < 1$, $|x| < \frac{1}{2}$.

We need to test the endpoints of the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

$\boxed{x = -\frac{1}{2}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(2)^n\left(-\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent,

Since $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergen p-series.

$\boxed{x = \frac{1}{2}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(2)^n\left(\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series.

Answer: $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(b) $\sum_{n=1}^{\infty} \frac{(-5)^n (x+2)^n}{4^n}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-5)^{n+1} (x+2)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-5)^n (x+2)^n} \right| = \lim_{n\to\infty} \frac{5}{4} |x+2| = \frac{5}{4}|x+2|$$

Setting $\frac{5}{4}|x+2| < 1$, we get $|x+2| < \frac{4}{5}$.

We need to test the endpoints of $\left(-2-\frac{4}{5}, -2+\frac{4}{5}\right)$.

$\boxed{x = -2 - \frac{4}{5}}$ $\sum_{n=1}^{\infty} \left(\frac{-5}{4}\right)^n \left(-2-\frac{4}{5}+2\right)^n = \sum_{n=1}^{\infty} \left(\left(\frac{-5}{4}\right)\left(\frac{-4}{5}\right)\right)^n = \sum_{n=1}^{\infty} 1.$

This series diverges by ~~te~~ test for divergence.

$\boxed{x = -2 + \frac{4}{5}}$ $\sum_{n=1}^{\infty} \left(\frac{-5}{4}\right)^n \left(-2+\frac{4}{5}+2\right)^n = \sum_{n=1}^{\infty} (-1)^n$, which also fails the test for divergence

Answer: $\left(\frac{-14}{5}, \frac{-6}{5}\right)$

BONUS QUESTION: (1 point each)

Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^A x^n}{n^B}.$$

The A, B given below are not unique.

For each of the intervals below, find an A and a B (A and B could be constants or functions of n) so that the given interval is the interval of convergence of the series.

(a) $(-1, 1)$ $A =$ n $B =$ 0

(b) $[-1, 1)$ $A =$ 1 $B =$ 1

(c) $(-1, 1]$ $A =$ n $B =$ 1

(d) $[-1, 1]$ $A =$ 1 $B =$ 2

(e) $\{0\}$ $A =$ 1 $B =$ -n

(f) $(-\infty, \infty)$ $A =$ 1 $B =$ n