Math 143

Midterm 1

October 11, 2018

Name: Key

UR ID: ______________________

Circle your Instructor's Name:

Kalyani Madhu   Ian Alevy

PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________

IMPORTANT: This exam has 8 questions, but you only need to answer 7. Below, circle the question you wish to **skip**. Then put an X on the question or in the answer box if that question has one.

1   2   3   4   5   6   7   8

1. (8 points)

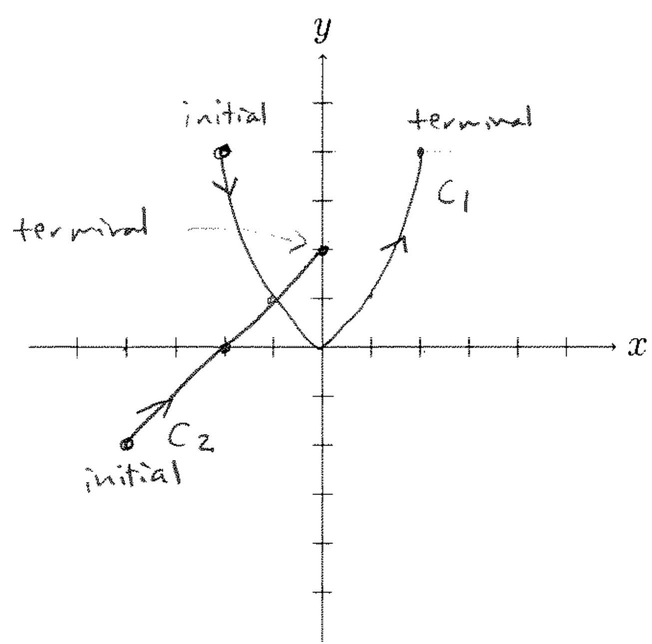
Two particles move in the plane. The first moves along the curve C_1 defined by the parametric equations

$$x = t,\ y = t^2,\ -2 < t < 2 \qquad y = x^2$$

and the second moves along the curve C_2 defined by the parametric equations

$$x = t - 2,\ y = t,\ -2 < t < 2 \qquad y = x + 2$$

(a) Plot both curves. Label them C_1 and C_2. For both curves label the initial and terminal points, and provide arrows to show in which direction the curve is being sketched.



(b) Do the particles collide? At what time or times? Justify your answer.

$\boxed{\text{They do not collide.}}$

If they collide, it is when $x^2 = x+2$,

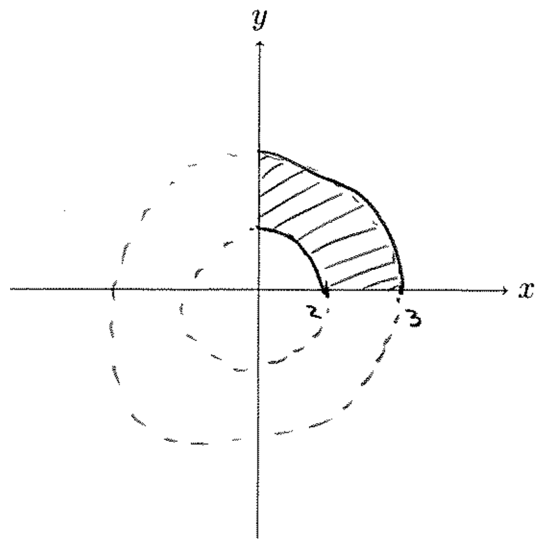
that is at $x = 2$ or at $x = -1$.

when $x=2$, Curve C_1 has $t=2$. Curve C_2 has $t = 4$ } not a collision point

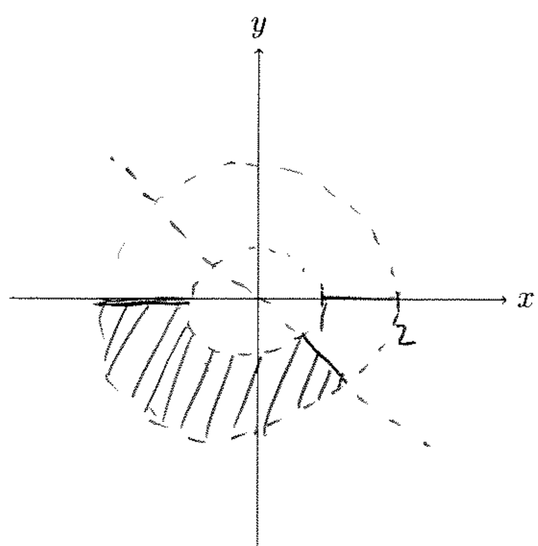
when $x=-1$ Curve C_1 has $t=-1$ Curve C_2 has $t=1$ } not a collision point.

2. (8 points) Sketch the region of the plane whose polar coordinates satisfy the given conditions

(a) $2 \leq r \leq 3$ and $0 \leq \theta \leq \frac{\pi}{2}$



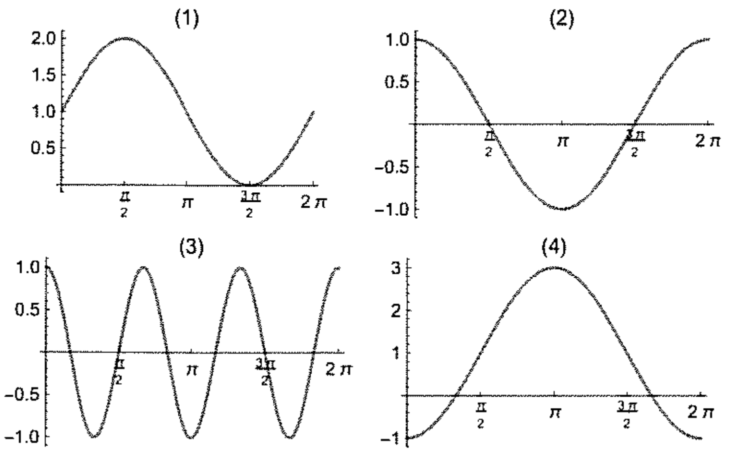
(b) $-2 < r < -1$ and $0 \leq \theta \leq \frac{3\pi}{4}$



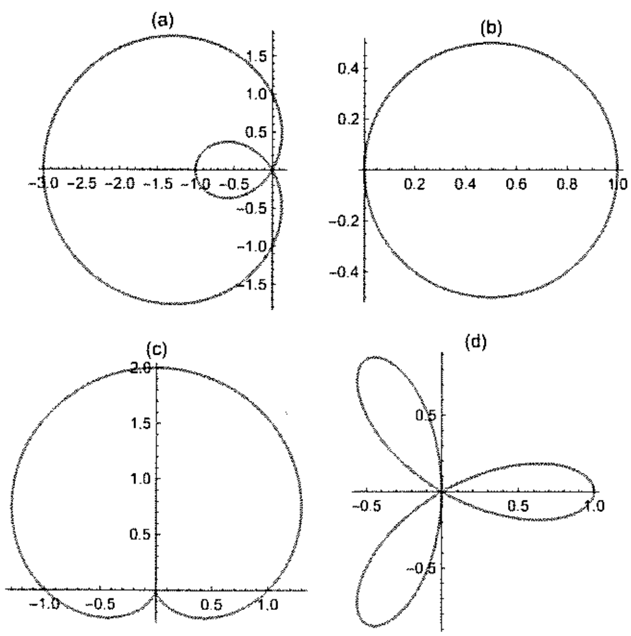
3. (8 points)

Match the graphs of functions sketched in the $r\theta$ plane to their polar plots in the xy-plane.

$r\theta$-plane:



xy-plane:



Answers:

1.) C   2.) B   3.) D   4.) A

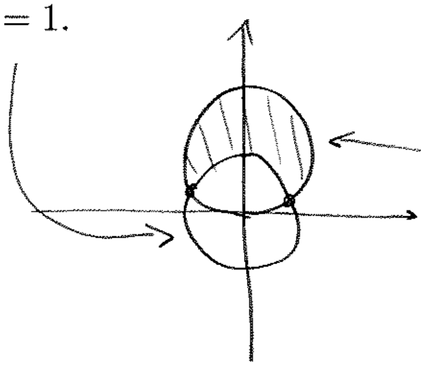
4. (8 points)

(a) Set up, but do not evaluate, an integral that gives the area of the region that lies inside the polar curve $r = 2\sin\theta$ and outside the polar curve $r = 1$.

$\to r^2 = 2r\sin\theta$

$x^2 + y^2 = 2y$

$x^2 + (y-1)^2 = 1$



Intersection pts:

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Answer:

$$\int_{\pi/6}^{5\pi/6} \frac{(2\sin\theta)^2 - 1^2}{2}\, d\theta = \int_{\pi/6}^{5\pi/6} \frac{4\sin^2\theta - 1}{2}\, d\theta$$

(b) Find the slope of the tangent line to the polar curve $r = 2\sin\theta$ when $\theta = \pi/6$.

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$= \frac{2\cos\theta\sin\theta + 2\sin\theta\cos\theta}{2\cos\theta\cos\theta - 2\sin\theta\sin\theta}$$

$$= \frac{2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\frac{3}{4} - \frac{1}{4}}$$

Answer: $\sqrt{3}$

$r' = 2\cos\theta$

$y = r\sin\theta = 2\sin^2\theta$

$\frac{dy}{d\theta} = 4\sin\theta\cos\theta$

$x = r\cos\theta = 2\sin\theta\cos\theta$

$\frac{dx}{d\theta} = 2\sin\theta(-\sin\theta) + 2\cos\theta\cos\theta$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{2(\cos^2\theta - \sin^2\theta)}$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \sqrt{3}$$

5. (8 points) (This question has 4 parts and is on two pages.) Determine whether each sequence converges or diverges. If it converges, find its limit. Justify your answers.

(a) $\left\{ \dfrac{\sin(4n+5)+7n}{5n} \right\}_{n=1}^{\infty}$

$$\frac{-1+7n}{5n} \le \frac{\sin(4n+5)+7n}{5n} \le \frac{1+7n}{5n}$$

Since $\lim\limits_{n\to\infty} \dfrac{-1+7n}{5n} = \lim\limits_{n\to\infty} \dfrac{1+7n}{5n} = \dfrac{7}{5}$, we get

Answer: $\dfrac{7}{5}$

(b) $\left\{ \dfrac{\ln(n)}{\ln(2n)} \right\}_{n=1}^{\infty}$ This is a question from the homework.

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(2n)} = \lim_{n\to\infty} \frac{\ln(n)}{\ln 2 + \ln(n)} = 1$$

Answer: 1

Question (5), second page

(c) $\left\{ \frac{\arctan(n)}{n} \right\}_{n=1}^{\infty}$

$$\frac{-\pi}{2n} < \frac{\arctan n}{n} \leq \frac{\pi}{2n}$$

Using the Squeeze Theorem,

we get $\frac{\arctan n}{n} \to 0$ as $n \to \infty$.

Answer: 0

(d) $\{\sin(\pi n)\}_{n=1}^{\infty}$

$\sin(\pi n) = 0$ for all n, so this is the constant sequence $\{0\}$.

Answer: 0

6. (8 points) (This question has two parts and is on two pages.)

(a) A sequence $\{a_n\}$ is defined recursively by the conditions

$$a_1 = 1 \qquad a_2 = 1 \qquad a_3 = 2 \qquad a_n = 2a_{n-1} + a_{n-3} \qquad \text{for} \qquad n \geq 4.$$

Find the terms $\{a_3, a_4, a_5, a_6, a_7\}$ in the sequence.

$a_4 = 2a_3 + a_1 = 5$

$a_5 = 2a_4 + a_2 = 10 + 1 = 11$

$a_6 = 2a_5 + a_3 = 22 + 2 = 24$

$a_7 = 2a_6 + a_4 = 48 + 5 = 53$

Answer: 2, 5, 11, 24, 53

(Question (6), second page)

(b) Is the sequence

$$a_n = \frac{n}{2^n}$$

monotonic for $n \geq 2$? If it is, is it increasing or decreasing? Justify your answer.

This sequence is decreasing.

① let $f(x) = \frac{x}{2^x}$. Then $a_n = f(n)$.

$$f'(x) = \frac{2^x - x2^x \ln 2}{(2^x)^2} = \frac{1 - x\ln 2}{2^x} = \frac{1 - \ln 2^x}{2^x}$$

Since $\ln 2^x \geq \ln 4$ if $x \geq 2$,

and $\ln 4 > \ln e = 1$, $f'(x) < 0$ for $x \geq 2$.

Since $f(x)$ is decreasing, so is $\{a_n\}$.

② $$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{2^n(n+1)}{2^{n+1}(n)} = \frac{n+1}{2n} < 1 \text{ for } n \geq 2.$$

Hence $\{a_n\}$ is decreasing.

7. (8 points)

(a) Suppose an infinite series $\sum_{n=1}^{\infty} a_n$ satisfies $\lim_{n\to\infty} S_n = 2$, where

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

i.) Find $\lim_{n\to\infty} a_n$, or state that there is not enough information.

Since S_n converges, by definition $\sum_{n=1}^{\infty} a_n$ converges.

So $\lim_{n\to\infty} a_n = 0$. (Test for Divergence.)

ii.) Find the value of the sum $\sum_{n=1}^{\infty} a_n$, or state that there is not enough information.

$\sum_{n=1}^{\infty} a_n = 2.$

(b) Suppose a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} a_n = 1$

i.) Does $\{a_n\}_{n=1}^{\infty}$ converge?

Yes, $\lim_{n\to\infty} a_n = 1$.

ii.) Does $\sum a_n$ converge?

No, as if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$,

Since $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ diverges.

8. (8 points) Determine whether or not the following series converge or diverge. If the series converges, provide its sum. Show your work.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{5^{n+1}}$ $= \frac{1}{5^2} + \frac{2}{5^3} + \frac{4}{5^4} + \ldots$

This is geometric with $a = \frac{1}{5^2}$

and $r = \frac{2}{5}$. Since $\left|\frac{2}{5}\right| < 1$, the series

converges. Its sum is $\frac{a}{1-r} = \frac{\frac{1}{5^2}}{1-\frac{2}{5}} = \frac{1}{5^2} \cdot \frac{5}{3} = \frac{1}{15}$

Answer: $\frac{1}{15}$

(b) $\displaystyle\sum_{n=0}^{\infty} \left(e^n - e^{n+1}\right)$

$S_n = (1 - \cancel{e}) + (\cancel{e} - \cancel{e^2}) + (\cancel{e^2} - \cancel{e^3}) + \ldots + (e^n - e^{n+1})$

$= 1 - e^{n+1}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - e^{n+1} = -\infty$

Answer: Diverges

EXTRA PAGE. You may use this page if you run out of space. Be sure to label your problems on this page and also include a note on the original page telling the graders to look for your work here.