Math 143

Final

December 16, 2018

Name: Solutions

UR ID: ______

Circle your Instructor's Name:

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PLEASE COPY THE HONOR PLEDGE AND SIGN:
(Cursive is not required).
I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE: ______

Part A

1. (12 points)

A curve is described by the parametric equations

$$x = e^t \cos(t),\ y = e^t \sin(t),\ 0 \le t \le 2\pi.$$

(a) Compute $\frac{dy}{dx}$ as a function of t.

$dy/dt = e^t \sin(t) + e^t \cos(t) = e^t(\sin t + \cos t)$

$dx/dt = e^t \cos(t) - e^t \sin(t) = e^t(\cos(t) - \sin(t))$

$dy/dx = \dfrac{dy/dt}{dx/dt}$

Answer: $\dfrac{\sin t + \cos t}{\cos t - \sin t}$

(b) For which t-values in the range $[0, 2\pi]$ is the tangent line horizontal?

$0 = dy/dt = e^t(\sin t + \cos t) \Rightarrow \sin t = -\cos t$

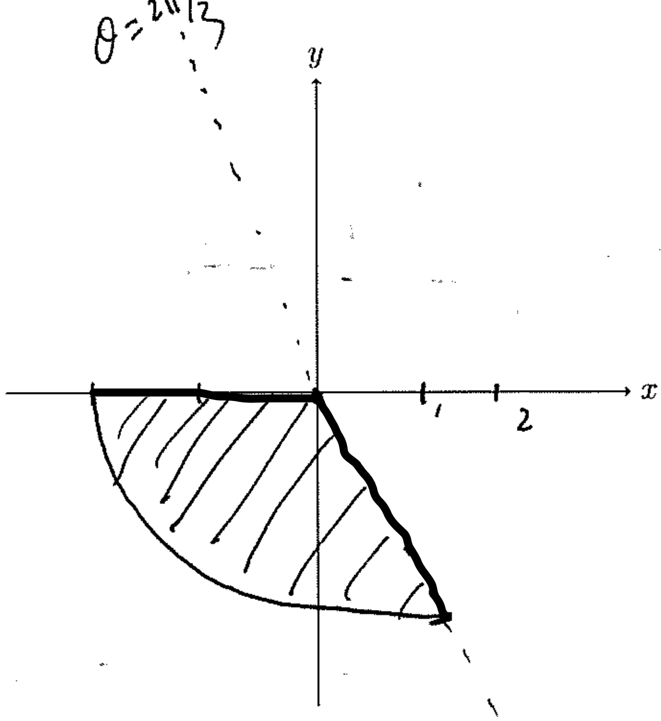
$dx/dt = e^t(\cos t - \sin t)$ is nonzero

Answer:
$t = 3\pi/4,\ 7\pi/4$

(c) Set up, but do not evaluate, an integral which computes the length of the curve.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\left(e^t(\cos(t) - \sin(t))\right)^2 + \left(e^t(\sin t + \cos t)\right)^2}\, dt$$

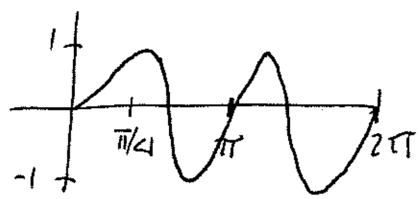
2. (7 points) On the axes below sketch the region of the plane whose polar coordinates satisfy the conditions $0 \le \theta \le 2\pi/3$ and $-2 \le r \le 0$.



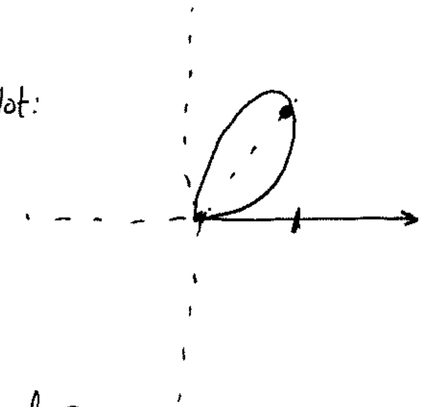
3. (10 points) Find the area enclosed by one loop of the polar curve $r = \sin 2\theta$. You may need to use the following trigonometric identity:

$$\sin^2(2\theta) = \frac{1 - \cos 4\theta}{2}.$$

$y = \sin 2x$



polar plot:



$$\text{area} = \int_0^{\pi/2} \frac{1}{2}(\sin 2\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 - \cos 4\theta \, d\theta = \frac{1}{4}\left(\theta - \frac{\sin 4\theta}{4}\right)\Bigg|_{\theta=0}^{\pi/2}$$

$$= \frac{1}{4} \cdot \pi/2 = \pi/8$$

Answer: $\pi/8$

4. (16 points) Determine whether each sequence converges or diverges. If it converges, check the appropriate box and write the limit in the space provided. If it diverges, check that box. **No work or justification is required, and no partial credit will be offered.**

(a) $\left\{\sqrt[n]{n^2+2}\right\}_{n=1}^{\infty}$

$\lim \sqrt[n]{n^2+2} = \lim e^{\frac{1}{n}\log(n^2+2)} \overset{L'H}{=} \lim e^{\frac{2n}{n^2+2}} = e^0 = 1$

☒ Converges to limit 1 ☐ Diverges

(b) $\left\{\dfrac{n!}{n(n-1)}\right\}_{n=1}^{\infty}$

$\dfrac{n!}{n(n-1)} = (n-2)!$

☐ Converges to limit ______ ☒ Diverges

(c) $\left\{3\cos\left(\dfrac{\pi}{2n}\right)+3\tan\left(\dfrac{\pi}{2n}\right)\right\}_{n=1}^{\infty}$

$\lim 3\cos\frac{\pi}{2n} + 3\tan\frac{\pi}{2n} = 3\cos 0 + 3\tan 0 = 3$

☒ Converges to limit 3 ☐ Diverges

(d) $\left\{\dfrac{2^{n+1}}{3+7^n}\right\}_{n=1}^{\infty}$ $\lim \dfrac{2^{n+1}}{3+7^n} = \lim 2\left(\dfrac{2^n}{7^n+3}\right) = \lim 2 \cdot \dfrac{(2/7)^n}{1+3/7^n} = 0$

☒ Converges to limit 0 ☐ Diverges

5. (20 points) For the following series, determine if the series is a geometric series, telescoping series, alternating series, p-series, or none of these. Check **all** categories which apply to the given series. Next, determine whether or not the series converges conditionally, converges absolutely, or diverges. **No work or justification is required, and no partial credit will be offered.**

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{2n+1}$ $\qquad \cos n\pi = (-1)^n$

Check ALL that apply:

☐ Geometric ☐ Telescoping ☒ Alternating ☐ p-series ☐ None of these

Check one:

☐ Converges absolutely ☒ Converges conditionally ☐ Diverges

(b) $\displaystyle\sum_{n=1}^{\infty} (-0.75)^n$

Check ALL that apply:

☒ Geometric ☐ Telescoping ☒ Alternating ☐ p-series ☐ None of these

Check one:

☒ Converges absolutely ☐ Converges conditionally ☐ Diverges

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Check ALL that apply:

☐ Geometric ☐ Telescoping ☐ Alternating ☒ p-series ☐ None of these

Check one:

☐ Converges absolutely ☐ Converges conditionally ☒ Diverges

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{3^{n+1}} = \sum \left(\frac{4^n}{3^n} * \frac{1}{3} \right)$

Check ALL that apply:

☒ Geometric ☐ Telescoping ☐ Alternating ☐ p-series ☐ None of these

Check one:

☐ Converges absolutely ☐ Converges conditionally ☒ Diverges

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n} = \frac{1}{2} - \frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \cdots$

Check ALL that apply:

☐ Geometric ☒ Telescoping ☐ Alternating ☐ p-series ☐ None of these

Check one:

☒ Converges absolutely ☐ Converges conditionally ☐ Diverges

6. (15 points)

(a) (10 pts) Use the integral test to show the convergence of

$$\sum_{\cancel{n=1}\; k=1}^{\infty} ke^{-k}.$$

$\int u\,dv = uv - \int v\,du$

$u = k \quad v = -e^{-k}$

$du = 1 \quad dv = e^{-k}$

(You do not need to check the hypotheses of the test.)

$$\int k e^{-k}\,dk = -e^{-k} k \Big|_1^{\infty} + \int_1^{\infty} e^{-k}\,dk$$

$$= e + (-e^{-k})\Big|_1^{\infty} = e + e = 2e$$

Since $2e$ is finite the series converges

(b) (5 pts) If we use

$$S_4 = \frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \frac{4}{e^4}$$

to estimate the sum of this series, what is the upper bound on the error **given by the integral test**?

$$\text{error} \le \int_4^{\infty} k e^{-k} = -e^{-k} k \Big|_4^{\infty} + \int_4^{\infty} e^{-k}\,dk$$

$$= 4e^{-4} + e^{-4} = 5e^{-4}$$

Answer: $R_4 \le \dfrac{5}{e^4}$

7. (10 points)

(a) Use the limit comparison test to determine if the following series converges or diverges. (Show your work and justify your answer.)

$$\sum_{n=3}^{\infty} \frac{\sqrt[3]{n-2}}{4n+7}$$

$\frac{n^{1/3}}{n} = n^{-2/3}$ compare with $\sum \frac{1}{n^{2/3}}$

which diverges by p-series test

$\lim_{n\to\infty} \frac{\sqrt[3]{n-2}}{4n+7} \cdot n^{2/3} = \lim_{n\to\infty} \frac{(n^3 - n^2 2)^{1/3}}{4n+7} = \frac{1}{4}$

Since $1/4 > 0$ the series ~~converges~~ diverges

Answer: diverges ~~converges~~

(b) Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3(2^n)}{7^{n+1}}. = \frac{3}{7} \sum_{n=0}^{\infty} \left(\frac{-2}{7}\right)^n = \frac{3}{7}\left(1 - \frac{2}{7} + \cdots\right)$$

$$= \frac{3}{7}\left(\frac{1}{1 - -2/7}\right) = \frac{3}{7} \cdot \frac{1}{9/7} = \frac{3}{7} \cdot \frac{7}{9} = \frac{3}{9} = 1/3$$

Answer: $1/3$

8. (10 points) Determine whether or not each of the following series converges or diverges. If it converges, determine if it converges conditionally or absolutely. Show your work, and say what test you used.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{(n^2+2)^{n+2}}{(n!)^n}$ use the ratio test

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}((n+1)^2+2)^{n+1+2}}{((n+1)!)^{n+1}} \cdot \frac{(n!)^n}{(-1)^n(n^2+2)^{n+2}}\right| = \lim \frac{((n+1)^2+2)^{n+3}}{(n^2+2)^{n+2}}\left(\frac{n!}{(n+1)!}\right)^n \frac{1}{n+1}$$

$$= \lim \frac{((n+1)^2+2)^{n+3}}{(n^2+2)^{n+2}} * \left(\frac{1}{n+1}\right)^{n+1} = \lim\left(\frac{(n+1)^2+2}{n^2+2}\right)^{n+2}((n+1)^2+2) * \left(\frac{1}{n+1}\right)^{n+1} = 0$$

Answer: Converges absolutely

(b) $\sum_{n=0}^{\infty}\frac{2^{n+2}((2n)!)}{n!((2n-1)!)}$

$$= \sum_{n=0}^{\infty}\frac{2^{n+2}}{n!}(2n)$$

ratio test $\lim_{n\to\infty}\left|\frac{2^{n+1+2}}{(n+1)!}2(n+1) * \frac{n!}{2^{n+2}(2n)}\right|$

$$= \lim_{n\to\infty}\frac{1}{n+1}\frac{(n+1)}{n}2 = 0$$

Answer: Converges absolutely

Part B

1. (20 points) Find the first four non-zero terms of the power series centered at 0 for the following functions and its radius of convergence.

(a) $f(x) = \frac{2}{4-3x^2}$ $= \frac{2}{4}\left(\frac{1}{1-\frac{3}{4}x^2}\right) = \frac{2}{4}\left(1 + \frac{3}{4}x^2 + \left(\frac{3}{4}x^2\right)^2 + \left(\frac{3}{4}x^2\right)^3 \ldots\right)$

$= \frac{2}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}x^2\right)^n$. Converges if $\left|\frac{3}{4}x^2\right| < 1 \Rightarrow |x| < \sqrt{\frac{4}{3}}$

$|x| < \frac{2}{\sqrt{3}}$

Power series: $\frac{1}{2}\left(1 + \frac{3}{4}x^2 + \left(\frac{3}{4}x^2\right)^2 + \left(\frac{3}{4}x^2\right)^3\right)$

Radius: $2/\sqrt{3}$

(b) $f(x) = \ln(x+2)$ $= \ln\left(2\left(1+x/2\right)\right) = \ln 2 + \ln\left(1 + x/2\right)$

$= \ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x/2)^n$ Converges if $|x/2| < 1 \Rightarrow |x| < 2$

$= \ln(2) + \frac{x}{2} - \left(\frac{x}{2}\right)^2\frac{1}{2} + \frac{1}{3}\left(\frac{x}{2}\right)^3$

Power series: $\ln 2 + \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{3}\left(\frac{x}{2}\right)^3$

Radius: 2

2. (20 points)

(a) Find the first 4 non-zero terms of the Taylor series for $\sin x$ at $\pi/6$.

$f(x) = \sin x \qquad f(\pi/6) = 1/2$

$f'(x) = \cos x \qquad f'(\pi/6) = \sqrt{3}/2$

$f''(x) = -\sin x \qquad f''(\pi/6) = -1/2$

$f'''(x) = -\cos(x) \qquad f'''(\cancel{3}\pi/6) = -\sqrt{3}/2$

Taylor series: $\frac{1}{2} + \frac{\sqrt{3}}{2} * (x - \pi/6) - \frac{1}{2} \cdot \frac{(x-\pi/6)^2}{2} - \frac{\sqrt{3}}{2} \frac{(x-\pi/6)^3}{3!}$

(b) Find the first 4 non-zero terms of the Maclaurin series for $\sqrt{1+2x}$.

$$\sqrt{1+2x} = 1 + \frac{1}{2}(2x) + \frac{(1/2)(1/2-1)}{2!}(2x)^2 + \frac{(1/2)(1/2-1)(1/2-2)}{3!}(2x)^3$$

$$= 1 + x - \frac{x^2}{2} + \frac{x^3}{2}$$

3. (30 points) Find the sum of the following series.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}$ $= (-1) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n4^n} =$

$= -\ln(1 + 1/4) = -\ln(5/4)$

Answer: $-\ln(5/4)$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ $= e^{-1}$

Answer: $1/e$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{4^n (2n)!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\sqrt{\frac{\pi}{4}}\right)^{2n} = \cos \sqrt{\frac{\pi}{4}}$

Answer: $\cos \sqrt{\pi/4}$

4. (30 points) Consider the function

$$f(x) = \frac{\sin x - x + \frac{1}{6}x^3}{2x^5}.$$

(a) Find the first 3 non-zero terms of a power series for the function.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$f(x) = \frac{\sin x - x + \frac{1}{6}x^3}{2x^5} = \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}}{2x^5} = \frac{1}{2\cdot 5!} - \frac{1}{2\cdot 7!}x^2 + \frac{1}{2\cdot 9!}x^4$$

(b) Find the first 3 non-zero terms for a series that represents $\int_0^1 f(x)dx$.

$$\int_0^1 f(x)dx = \int_0^1 \frac{1}{2\cdot 5!} - \frac{1}{2\cdot 7!}x^2 + \frac{1}{2\cdot 9!}x^4 \, dx$$

$$= \left(\frac{1}{2\cdot 5!}x - \frac{1}{2\cdot 7!\cdot 3}x^3 + \frac{1}{2\cdot 9!\cdot 5}x^5\right)\Big|_{x=0}^{1}$$

$$= \frac{1}{2\cdot 5!} - \frac{1}{2\cdot 3\cdot 7!} + \frac{1}{2\cdot 5\cdot 9!}$$

(c) Find $\lim_{x\to 0} f(x)$.

$$\lim_{x\to 0} f(x) = \frac{1}{2\cdot 5!} = 1/240$$

EXTRA SPACE. Use this space if you run out elsewhere. Be sure to label your problems and also include a note on the original page telling the graders to look for your work here.