Math 143: Calculus III

Midterm II

November 21st, 2017

NAME (please print legibly): ______________________________
Your University ID Number: ______________________________
Your University email ______________________________
Please circle your section:

Tucker TR 2:00pm **Yamazaki MW 9:00am**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 8 pages.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	20	
4	20	
5	20	
6	10	
7	10	
TOTAL	100	

1. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use.*

$$\sum_{n=1}^{\infty} \frac{2n\cos(n)}{n^3+7}$$

As

$$\left|\frac{2n\cos(n)}{n^3+7}\right| \le \frac{2n}{n^3+7} \le \frac{2}{n^2},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test, we see that

$$\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges which implies by CT that $\sum_{n=1}^{\infty} \frac{2n\cos(n)}{n^3+7}$ converges absolutely.

2. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use.*

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+10}$$

$\frac{\sqrt{n}}{n+10}$ is positive, decreasing to 0. Hence, by AST, $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+10}$ converges. But $\lim_{n\to\infty} \frac{(\frac{\sqrt{n}}{n+10})}{(\frac{1}{\sqrt{n}})} = \lim_{n\to\infty} \frac{n}{n+10} = 1$ so that as $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = +\infty$ by p-test, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+10} = \sum_{n=1}^{\infty} |\frac{(-1)^n\sqrt{n}}{n+10}|$ also diverges by LCT. Thus, conditional convergence.

3. (20 points) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-3)^n (x+1)^n}{2^n \sqrt{n}}.$$

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1}(x+1)^{n+1}}{2^{n+1}\sqrt{n+1}} \frac{2^n\sqrt{n}}{(-3)^n(x+1)^n} \right| = |x+1| \lim_{n\to\infty} \frac{3}{2}\frac{\sqrt{n}}{\sqrt{n+1}} = |x+1|\frac{3}{2} < 1 \text{ if } |x+1| < \frac{2}{3}.$$

Hence, ROC is $\frac{2}{3}$. From $(-\frac{5}{3}, -\frac{1}{3})$, we check that if $x = -\frac{5}{3}$, then $\sum_{n=1}^{\infty} \frac{(-3)^n(-\frac{2}{3})^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-test, whereas if $x = -\frac{1}{3}$, then $\sum_{n=1}^{\infty} \frac{(-3)^n(\frac{2}{3})^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST as $\frac{1}{\sqrt{n}}$ is positive, and decreasing to 0. Thus, IOC is $(-\frac{5}{3}, -\frac{1}{3}]$.

4. (20 points) Consider the function $f(x) = e^{-x}$.

(a) Find a power series expansion of $f(x)$ about $x = -4$. Write out the first three nonzero terms, and express the series in sigma notation.

$$c_0 = \frac{e^4}{0!},$$
$$c_1 = \frac{-e^4}{1!},$$
$$c_2 = \frac{e^4}{2!}$$

from which we see that $c_n = \frac{(-1)^n e^4}{n!}$. Thus,

$$\begin{aligned} e^{-x} =& \sum_{n=0}^{\infty} \frac{(-1)^n e^4}{n!}(x+4)^n \\ =& e^4 - e^4(x+4) + \frac{e^4}{2}(x+4)^2 - \dots \end{aligned}$$

(b) Use the ratio test to find the radius and interval of convergence of the series you found in (a). *No credit will be given for solutions not using the ratio test.*

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}e^4(x+4)^{n+1}}{(n+1)!} \frac{n!}{(-1)^n e^4 (x+4)^n}\right| = |x+4| \lim_{n\to\infty} \frac{1}{n+1} = 0$$

and thus ROC is $+\infty$ while IOC is $(-\infty, \infty)$.

5. (20 points)

(a) Find the Maclaurin series expansion of the function

$$f(x) = \frac{x^2 - \sin(x^2)}{x^6}.$$

Write out the first four nonzero terms, and express the series in sigma notation.

$$\begin{aligned}
f(x) =& \frac{x^2 - \sin(x^2)}{x^6} \\
=& \frac{x^2 - \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}}{x^6} \\
=& \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-4}}{(2n+1)!} \\
=& \frac{(-1)^2 x^0}{3!} + \frac{(-1)^3 x^4}{5!} + \frac{(-1)^4 x^8}{7!} + \frac{(-1)^5 x^{12}}{9!} + \ldots
\end{aligned}$$

(b) What is the value of $f^{(12)}(0)$?

As $\frac{f^{(12)}(0)}{12!} x^{12} = \frac{(-1)^5 x^{12}}{9!}$, we have $f^{(12)}(0) = \frac{(-1)^5 12!}{9!}$.

(c) What is the value of $f^{(11)}(0)$?

$f^{(11)}(0) = 0$.

(d) What is the value of $\lim_{x \to 0} f(x)$?

$$\begin{aligned}
\lim_{x \to 0} f(x) =& \lim_{x \to 0} \frac{(-1)^2 x^0}{3!} + \frac{(-1)^3 x^4}{5!} + \ldots \\
=& \frac{1}{6}.
\end{aligned}$$

6. (10 points) Write out the first two terms and then find the sum of each of the following convergent series. *You do not need to show the series are convergent. Your table of Maclaurin series expansions might be helpful.*

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(-3)^n} =$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(-3)^n} = \frac{(-1)^0}{1(-3)^1} + \frac{(-1)^1}{2(-3)^2} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-\frac{1}{3})^n = \ln(\frac{2}{3}).$$

(b) $\sum_{n=0}^{\infty} \frac{4^n}{(-5)^n n!} =$

$$\sum_{n=0}^{\infty} \frac{4^n}{(-5)^n n!} = \frac{4^0}{(-5)^0 0!} + \frac{4!}{(-5)^1 1!} + \ldots = e^{-\frac{4}{5}}.$$

(c) $\sum_{n=0}^{\infty} \frac{3(-1)^{n-1}}{(2n+1)2^{2n+1}} =$

$$\sum_{n=0}^{\infty} \frac{3(-1)^{n-1}}{(2n+1)2^{2n+1}} = \frac{3(-1)^{-1}}{2} + \frac{3(-1)^0}{(3)2^3} + \ldots = (-3)\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(\frac{1}{2})^{2n+1} = (-3)\tan^{-1}(\frac{1}{2}).$$

7. (10 points) Consider the parametric equations for a curve $C(\theta)$ defined by

$$x = 5\cos(\theta), \qquad y = 2\sin(\theta)$$

(a) Eliminate the parameter, and write the resulting Cartesian equation in the form given below. *No credit will be given for solutions not showing any work.*

$$\frac{y^2}{4} = \sin^2(\theta) = 1 - \cos^2(\theta) = 1 - (\frac{x}{5})^2.$$

(b) Find an interval of θ-values so that $C(\theta) = (5\cos(\theta), 2\sin(\theta))$ traces out the upper half of an ellipse (in the counter-clockwise direction).

$[0, \pi]$.