Math 143: Calculus III

Midterm I

October 17th, 2017

NAME (please print legibly): SOLUTIONS
Your University ID Number:
Your University email
Please circle your section:

Tucker TR 2:00pm **Yamazaki MW 9:00am**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature:

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 8 pages.**

QUESTION	VALUE	SCORE
1	15	
2	15	
3	20	
4	10	
5	10	
6	10	
7	20	
TOTAL	100	

1. (15 points) Determine whether the following sequences converge or diverge. If they converge, find their limit. If they diverge, state whether they diverge to $+\infty$, $-\infty$ or because they oscillate. **Justify and show all your work.**

(a)

$$a_n = \frac{(-e)^n}{n}$$

$\boxed{\text{DIV OSCL}}$

$\lim_{n\to\infty} \left|\frac{(-e)^n}{n}\right| = \infty$ but even terms are + and odd terms are −

$\left|\frac{(-e)^n}{n}\right| = \frac{e^n}{n}$ & $\lim_{n\to\infty} \frac{e^n}{n} = \lim_{x\to\infty} \frac{e^x}{x} \underset{\uparrow}{=} \lim_{x\to\infty} e^x = \infty$

$\frac{\infty}{\infty}$ l'HR

(b)

$$a_n = \left(1 + \frac{1}{2n}\right)^n$$

$\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^n = \lim_{2n\to\infty} \left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{1}{2}} = \left(\lim_{2n\to\infty} \left(1+\frac{1}{2n}\right)^{2n}\right)^{1/2} = \boxed{e^{1/2}}$

since $n\to\infty \iff 2n\to\infty$

Cont's fxn thm.

$\boxed{\text{CONV}}$

(c)

$$a_n = \cos\left(\frac{\ln(n)}{n}\right)$$

$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

$\lim_{n\to\infty} \cos\left(\frac{\ln(n)}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\ln(n)}{n}\right) = \cos(0) = \boxed{1}$

Cont's fxn 2 thm.

$\boxed{\text{CONV}}$

2. (15 points) Determine whether the following series converge or diverge. If a series converges, find its sum. **Justify and show all your work. Name any test you are using.**

(a)

$$\sum_{n=0}^{\infty} \frac{1+2^n}{3^n} \overset{\text{because both are conv.}}{=} \sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=0}^{\infty} \frac{2^n}{3^n}$$

geom. $\boxed{a=1,\ r=1/3}$ geom. $\boxed{a=1,\ r=2/3}$

$$= \frac{1}{1-1/3} + \frac{1}{1-2/3} = 1/\tfrac{2}{3} + 1/\tfrac{1}{3} = \frac{3}{2} + 3 = \boxed{4.5}$$

$\boxed{\text{CONV}}$ because sum of convergent series is conv. and both are conv. geom. series by GST

(b)

$$\sum_{n=1}^{\infty} \frac{7^{2n}}{24^{n+1}}$$

$$= \frac{1}{24} \sum_{n=1}^{\infty} \left(\frac{7^2}{24}\right)^n = \frac{1}{24} \sum_{n=1}^{\infty} \left(\frac{49}{24}\right)^n = \boxed{\text{DIV}}$$ by GST with $|r| = \frac{49}{24} > 1$

(c)

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)$$ telescoping!

$$S_k = \cos(1) - \cos\left(\tfrac{1}{2}\right) + \cos\left(\tfrac{1}{2}\right) - \cos\left(\tfrac{1}{3}\right) + \cdots + \cos\left(\tfrac{1}{k}\right) - \cos\left(\tfrac{1}{k+1}\right)$$

$$= \cos(1) - \cos\left(\tfrac{1}{k+1}\right)$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \left[\cos(1) - \cos\left(\overbrace{\tfrac{1}{k+1}}^{0 \text{ as } k\to\infty}\right)\right] = \cos(1) - \cos(0) = \boxed{\cos(1) - 1}\ \boxed{\text{CONV}}$$

3. (20 points) Determine whether the following series converge or diverge. **Justify and show all your work. Name any test you are using.**

(a)

$$\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n}$$

CT:

① $\ln(n) \geq 1$ for $n \geq 3$ $\Rightarrow$ $(\ln(n))^2 \geq 1$ for $n \geq 3$

$\Rightarrow \dfrac{(\ln(n))^2}{n} \geq \dfrac{1}{n}$ for $n \geq 3$

② $\sum \frac{1}{n}$ DIV by p-test with $p = 1$

③ By CT, $\sum \dfrac{(\ln(n))^2}{n}$ DIV as well

(b)

$$\sum_{n=1}^{\infty} \frac{3^n + 1}{n + 2^n}$$

Test for divergence $\quad \lim_{n\to\infty} \dfrac{3^n+1}{n+2^n} \underset{\frac{\infty}{\infty},\ l'HR}{=} \lim_{n\to\infty} \dfrac{\ln(3)\cdot 3^n}{1+\ln(2)2^n} \underset{\frac{\infty}{\infty}\ l'HR}{=} \lim_{n\to\infty} \dfrac{(\ln(3))^2\, 3^n}{(\ln(2))^2\, 2^n} = \infty \neq 0$ $\boxed{\text{DIV by Test for DIV}}$

$(\frac{3}{2})^n \to \infty$ as $n \to \infty$

[alternate solution: $\lim_{n\to\infty} \dfrac{3^n+1}{n+2^n} = \lim_{n\to\infty} \dfrac{(\frac{3}{2})^n + \frac{1}{2^n}}{\frac{n}{2^n} + 1} = \infty \neq 0$ $\boxed{\text{DIV}}$ by test for DIV

divide numerator and denominator by 2^n

$\frac{1}{2^n} \to 0$

$\frac{n}{2^n} \to 0$ by l'HR

$(\frac{3}{2})^n \to \infty$ as $n \to \infty$

(note: limit comparison test with $\sum \frac{3^n}{2^n}$ works, too!)

4. (10 points) Use the integral test to determine whether the following series converges or diverges. **To get full credit you must use the integral test.**

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u^2}\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty}\Big[-u^{-1}\Big]_{\ln(2)}^{\ln(b)} = \lim_{b\to\infty}\left[\cancel{\frac{-1}{\ln(b)}} + \frac{1}{\ln(2)}\right] = \boxed{\frac{1}{\ln(2)}} < \infty$$

0 as $b \to \infty$

CONV integral

So, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ $\boxed{\text{CONV}}$ as well.

5. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{2\sin(n)\sqrt{n}}{n^2+5} \qquad \boxed{\text{ABS. CONV.}}$$

Test for abs. conv. first: $\displaystyle\sum_{n=1}^{\infty} \left| \frac{2\sin(n)\sqrt{n}}{n^2+5} \right|$ has all positive terms

① $|\sin(n)| \le 1 \Rightarrow \left| \frac{2\sin(n)\sqrt{n}}{n^2+5} \right| \le \frac{2\sqrt{n}}{n^2+5}$ so by CT if $\sum \frac{2\sqrt{n}}{n^2+5}$ conv.

then so does $\sum \left| \frac{2\sin(n)\sqrt{n}}{n^2+5} \right|$

② ⓐ $\displaystyle\lim_{n\to\infty} \frac{2\sqrt{n}}{n^2+5} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{2n^2}{n^2+5} = 2 \neq 0$ nonzero const.

ratio of lead coeff

ⓑ $\sum \frac{1}{n^{3/2}}$ conv by p-test

ⓒ So $\sum \frac{2\sqrt{n}}{n^2+5}$ conv by LCT (by ⓐ and ⓑ)

No need to test for cond. conv. since $\sum \frac{2\sin(n)\sqrt{n}}{n^2+5}$ is $\boxed{\text{ABS CONV}}$ by CT & LCT

6. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

① $\sum \frac{\ln(n)}{n}$ DIV by CT since $\sum \frac{1}{n}$ DIV and $\ln(n) > 1$ for $n \geq 3$

So $\frac{\ln(n)}{n} > \frac{1}{n}$ for $n \geq 3$ so $\sum \frac{(-1)^n \ln(n)}{n}$ is NOT Abs. Conv.

② $\sum \frac{(-1)^n \ln(n)}{n}$ is alternating ✓

decreasing ✓ $f(x) = \frac{\ln(x)}{x} \Rightarrow f'(x) = \frac{\frac{1}{x}x - \ln(x)}{x^2}$

$= \frac{1-\ln(x)}{x^2} < 0$ for $x \geq 3$

$\frac{\ln(n)}{n} \to 0$ as $n \to \infty$ by L'HR ✓

So the series in Conv. by AST

③ Thus, $\sum \frac{(-1)^n \ln(n)}{n}$ is $\boxed{\text{COND CONV}}$

7. (20 points)

(a) Use the ratio test to determine if the following series is absolutely convergent or divergent. **To get full credit you must use the ratio test.**

$$1 - \frac{2!}{1\cdot 3} + \frac{3!}{1\cdot 3\cdot 5} - \ldots + (-1)^{n-1}\frac{n!}{1\cdot 3\cdot 5\ldots(2n-1)} + \ldots$$

ratio test: $\lim_{n\to\infty}\left|\frac{(-1)^n(n+1)!}{\cancel{1\cdot3\cdot5\cdots(2n-1)}(2(n+1)-1)}\cdot\frac{\cancel{1\cdot3\cdot5\cdots(2n-1)}}{(-1)^{n-1}\,n!}\right| =$

$\lim_{n\to\infty}\left|\frac{-1(n+1)}{(2n+1)}\right| = \left|\frac{-1}{2}\right| = \frac{1}{2} < 1$ $\boxed{\text{ABS CONV}}$

(b) Use the root test to determine if the following series is absolutely convergent or divergent. **To get full credit you must use the root test.**

$$\sum_{n=1}^{\infty}\left(\frac{n2^n}{3^n}\right)^n.$$

root test: $\lim_{n\to\infty}\left|\left(\frac{n2^n}{3^n}\right)^{\cancel{n}}\right|^{\cancel{1/n}} = \lim_{n\to\infty}\frac{n2^n}{3^n} = \lim_{n\to\infty} n\left(\frac{2}{3}\right)^n$

$\boxed{\text{type } \infty\cdot 0}$

$= \lim_{n\to\infty}\frac{n}{\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty}\frac{1}{\ln\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^n} = 0 < 1$ so $\boxed{\text{ABS CONV}}$

$\frac{\infty}{\infty}$, l'HR

$\left(\frac{3}{2}\right)^n \to \infty$ as $n\to\infty$

$\left(\frac{2}{3}\right)^n = \frac{1}{\left(\frac{3}{2}\right)^n}$