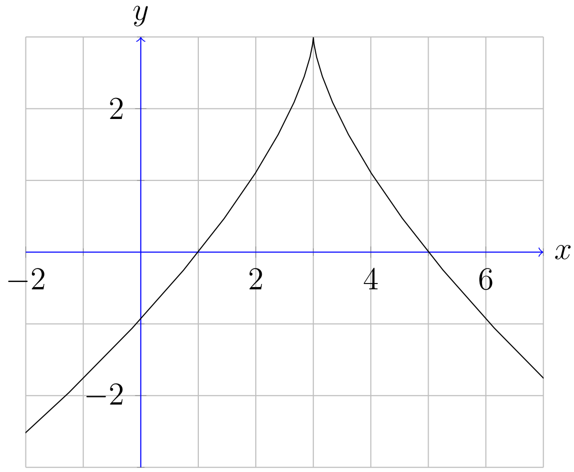
# Math 143: Calculus III

Midterm 1 ANSWERS

February 18, 2015

**1. (18 points)** Consider the parametric curve $x = 2t^3 + 3$, $y = 3 - 3t^2$, whose graph appears below:




(a) Find all points on the curve such that the slope of the tangent line is $\frac{1}{2}$.

•The slope of the tangent line is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-6t}{6t^2} = -\frac{1}{t}$. This equals $\frac{1}{2}$ when $t = -2$, which gives $(x, y) = \boxed{(-13, -9)}$.

(b) Find the coordinates of the two intersection points of the curve with the $x$-axis, and the corresponding values of $t$.

•These points occur when $y = 0$, meaning $3 - 3t^2 = 0$. Solving gives $t = \boxed{-1, 1}$ and plugging in yields $(x, y) = \boxed{(1, 0) \text{ and } (5, 0)}$.

(c) Find the area of the region lying below the curve and above the $x$-axis.

•Notice that $x$ is an increasing function of $t$, so the curve is traced left to right. Then the desired area is given by

$$\int_{-1}^{1} \left[3 - 3t^2\right] \cdot \left[6t^2\right] \, dt = 18 \int_{-1}^{1} \left[t^2 - t^4\right] \, dt = 18 \left[\frac{t^3}{3} - \frac{t^5}{5}\right]\Bigg|_{t=-1}^{1} = \boxed{18\left(\frac{2}{3} - \frac{2}{5}\right)} = \boxed{\frac{24}{5}}.$$

**2. (8 points)** Make the following coordinate conversions. (Your answers should not have any functions in them.)

(a) Convert the point $(x, y) = (-1, \sqrt{3})$ to polar coordinates.

•We have $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$ and $\theta = \tan^{-1}(-\sqrt{3}) + \pi = \frac{2\pi}{3}$, giving $\boxed{\left(2, \frac{2\pi}{3}\right)}$.

(b) Convert the point $(r, \theta) = (\sqrt{3}, \pi/6)$ to rectangular coordinates.

•We have $x = r\cos(\theta) = \frac{3}{2}$ and $y = r\sin(\theta) = \frac{\sqrt{3}}{2}$, giving $\boxed{\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)}$.

(c) Convert the point $(x, y) = (6, -6)$ to polar coordinates.

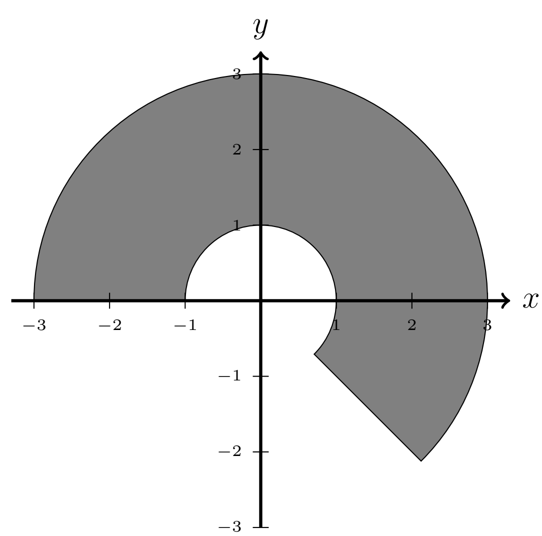
•We have $r = \sqrt{6^2 + (-6)^2} = 6\sqrt{2}$ and $\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$, giving $\boxed{\left(6\sqrt{2}, -\frac{\pi}{4}\right)}$.

(d) Convert the point $(r, \theta) = (-6, -5\pi)$ to rectangular coordinates.

•We have $x = r\cos(\theta) = 6$ and $y = r\sin(\theta) = 0$, giving $\boxed{(6, 0)}$.

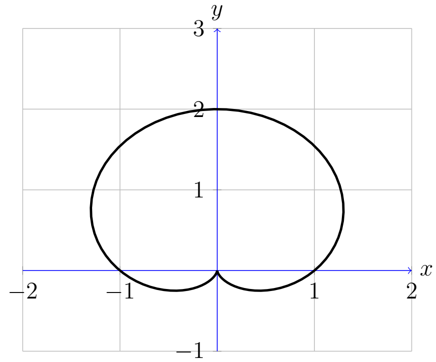
**3. (6 points)** Sketch the polar region defined by the inequalities $-\frac{\pi}{4} \leq \theta \leq \pi$, $1 \leq r \leq 3$. Make sure to label any relevant distances on the $x$ and $y$ axes.

- This is a sector of an annulus:

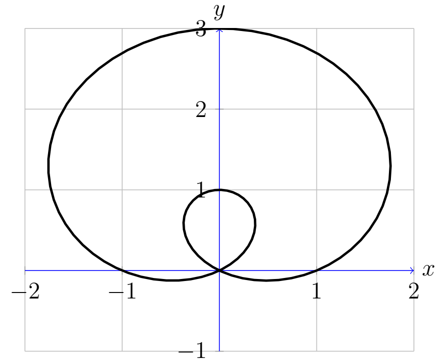


**4. (12 points)** Match the following curves to their equations.

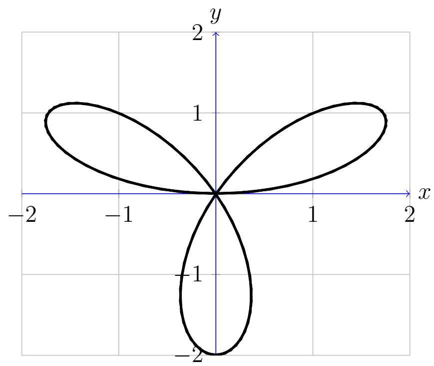
**Curves:**



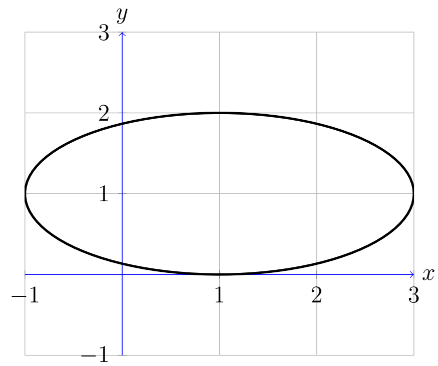

(a)



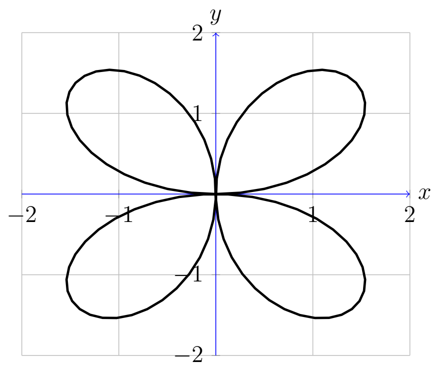

(b)



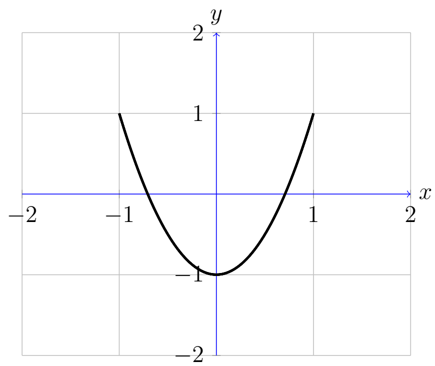

(c)




(d)




(e)




(f)

**Equations:**

1.) $x = \sin(t)$,
$y = 2\sin^2(t) - 1$

2.) $r = 2\sin(2\theta)$

3.) $r = 1 + 2\cos(\theta)$

4.) $x = \cos^2(t)$,
$y = 1 - 2\sin(t)$

5.) $r = 2\sin(3\theta)$

6.) $r = 1 + 2\sin(\theta)$

7.) $x = 1 + 2\cos(t)$,
$y = 1 + \sin(t)$

8.) $r = 2\sin(4\theta)$

9.) $r = 1 + \sin(\theta)$

**Answers:**

(a) 9 (b) 6 (c) 5 (d) 7 (e) 2 (f) 1

The simplest strategy is to plug in a few values of $t$ or $\theta$ to each equation.

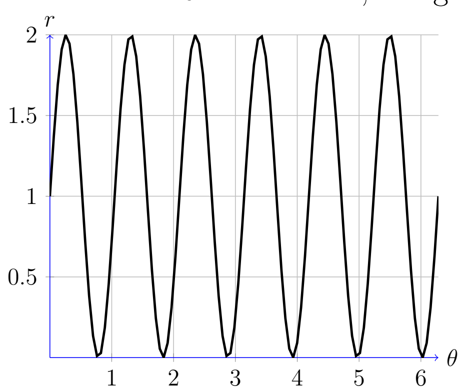
**5. (12 points)** Consider the polar curve $r = 1 + \sin(6\theta)$ for $0 \le \theta \le 2\pi$.

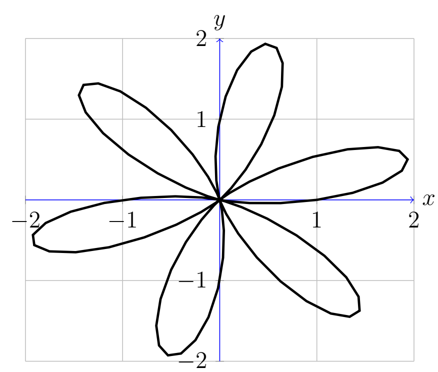
(a) Find all values of $\theta$ in the interval $[0, 2\pi]$ where this curve passes through the origin.

- The curve goes through the origin when $r = 0$, namely when $\sin(6\theta) = -1$. This occurs when $6\theta = \frac{3\pi}{2} + 2\pi k$ for some integer $k$: thus, for $\theta = \frac{\pi}{4} + \frac{\pi k}{3}$.
- The values in $[0, 2\pi]$ are $\theta = \boxed{\frac{\pi}{4} + \frac{\pi k}{3}}$ for $0 \le k \le 5$.
- Explicitly, they are $\theta = \boxed{\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{24}}$.

(b) Sketch the graph of this curve. Make sure to label any relevant distances on the $x$ and $y$ axes.

We first sketch $r = 1 + \sin(6\theta)$ as a rectangular plot. We then use the features of this graph to make the polar plot. We know that sine oscillates between $-1$ and 1 so the graph will start at $(0, 1)$ and move up to a maximum of 2 and down to a minimum of 0. Between $\theta = 0$ and $\theta = 2\pi$, the graph will go through six periods:




From this we see the polar curve will start at $(1, 0)$ and sweep out away from the origin to a maximum distance of $r = 2$, then back toward the origin at $\theta = \pi/4$ (as we computed in part (a)). It will then sweep out again, and then back in, passing through $(0, 1)$ at $\theta = \pi/2$, hitting the origin once more at $\theta = 7\pi/12$. It will then sweep out and back in again, symmetrically, until we reach $\theta = 2\pi$. This occurs six times, so the overall shape is of a tilted six-petaled flower (above).

Another option is to make a table of values of points $(r, \theta)$ on the graph, for various nice values of $\theta$. Although this would work in principle, it would require plugging in at least 10 or so points to get the proper shape.

**6. (12 points)** At time $t$ seconds ($t \geq 0$), a particle's position in the plane is given by

$$x = e^t + e^{-t}, \quad y = 2t + 5.$$

(a) Is the particle ever moving directly in the vertical direction? Directly in the horizontal direction? If so, when?

- The slope of the tangent line is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2}{e^t - e^{-t}}$. Notice that a slope of zero means the particle is moving horizontally, while a slope of $\infty$ means the particle is moving vertically.
- The numerator is never zero, and the denominator is always finite. So the slope is never zero, meaning that the particle is $\boxed{\text{never moving directly horizontally}}$.
- The slope is $\infty$ when the denominator is zero, which occurs when $e^t = e^{-t}$. So the particle is moving $\boxed{\text{directly vertically when } t = 0}$.

(b) Find the distance traveled by the particle between time $t = 0$s and time $t = 1$s.

- The arclength is

$$\begin{aligned} s &= \int_0^1 \sqrt{(e^t - e^{-t})^2 + 2^2}\, dt = \int_0^1 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\, dt \\ &= \int_0^1 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt = \int_0^1 \sqrt{(e^t + e^{-t})^2}\, dt \\ &= \int_0^1 (e^t + e^{-t})\, dt = \left(e^t - e^{-t}\right) \Big|_{t=-1}^{1} = \boxed{e - \frac{1}{e}} \end{aligned}$$

**7. (7 points)** Find the area enclosed by the polar curve $r = \sqrt{3 + 2\sin(4\theta)}$.

- A quick sketch of this curve indicates that it completes one revolution between $\theta = 0$ and $\theta = 2\pi$, and never crosses through the origin (because the quantity under the square root is always positive). It is not actually necessary to sketch the curve to see this.

- Thus the area is given by

$$A = \frac{1}{2} \int_0^{2\pi} r^2\, d\theta = \frac{1}{2} \int_0^{2\pi} (3 + 2\sin(4\theta))\, d\theta = \frac{1}{2} \left( 3\theta - \frac{1}{2}\cos(4\theta) \right) \Big|_{\theta=0}^{2\pi} = \boxed{3\pi}.$$

**8. (9 points)** Let $z = 2 - 2i$ and $w = 5 + i$. Compute:

(a) $z^2 + 2w$, in $a + bi$ form.

•We have $z^2 = (2-2i)^2 = 4 - 8i + 4i^2 = -8i$ so $z^2 + 2w = \boxed{10 - 6i}$.

(b) $\dfrac{z}{w}$, in $a + bi$ form.

•We write $\dfrac{2-2i}{5+i} = \dfrac{(2-2i)(5-i)}{(5+i)(5-i)} = \dfrac{10 - 10i - 2i + 2i^2}{25 - i^2} = \dfrac{8 - 12i}{26} = \boxed{\dfrac{4}{13} - \dfrac{6}{13}i}$.

(c) Real numbers $r$ and $\theta$ such that $z = r\,e^{i\theta}$.

•We take $r = |z| = \boxed{2\sqrt{2}}$ and $\theta = \tan^{-1}(-2/2) = \boxed{-\dfrac{\pi}{4}}$ (or $\boxed{\dfrac{7\pi}{4}}$).

**9. (8 points)** Find, in $a + bi$ form, all complex numbers $z$ such that $z^4 = -16$.

- We convert to polar form: $-16 = 16 \cdot e^{i\pi}$.
- Then the fourth roots are given by $16^{1/4} \cdot e^{i\pi/4} \cdot e^{2ki\pi/4}$ for $k = 0, 1, 2, 3$.
- These simplify to $2 \cdot \left( \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} i \right) \cdot i^k = (\sqrt{2} + \sqrt{2}i)i^k$.
- Thus, we obtain $\boxed{\sqrt{2} + \sqrt{2}i,\ \sqrt{2} - \sqrt{2}i,\ -\sqrt{2} + \sqrt{2}i,\ -\sqrt{2} - \sqrt{2}i}$.

**10. (8 points)** Two complex numbers have a sum of 4 and a product of 6. Find them.

- Suppose they are $z$ and $w$. Then $z + w = 4$ and $zw = 6$.
- Since $w = 4 - z$ we plug in to the second equation to get $z(4 - z) = 6$, or $4z - z^2 = 6$.
- This is the same as the quadratic equation $z^2 - 4z + 6 = 0$.
- Solving yields $z = \dfrac{4 \pm \sqrt{16 - 24}}{2} = \dfrac{4 \pm \sqrt{-8}}{2} = \dfrac{4 \pm 2\sqrt{2}i}{2} = 2 \pm \sqrt{2}i$.
- Hence the numbers are $\boxed{2 + \sqrt{2}i \text{ and } 2 - \sqrt{2}i}$.