Math 143: Calculus III

Midterm II

November 8th, 2016

Please circle your section:

Yamazaki MWF 9am Tucker TR 2pm

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Put your answers in the spaces provided.
- You are responsible for checking that this exam has all 8 pages.

QUESTION	VALUE	SCORE
1	15	
2	20	
3	15	
4	15	
5	20	
6	15	
TOTAL	100	

Common Taylor series centered at $x = 0$:

Function	Taylor Series	Initial Terms	Converges for
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + x^4 + \cdots$	$-1 < x < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$1 - x + x^2 - x^3 + x^4 - \cdots$	$-1 < x < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	All x
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\tan^{-1}(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$-1 \le x \le 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 < x \le 1$

1. (15 points) Use the root test or ratio test to determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

(a)

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{26^n n^2}$$ ratio test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^{n+1}}{26^{n+1}(n+1)^2}\cdot\frac{26^n\cdot n^2}{(-5)^n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^{n+1}}{(-5)^n}\cdot\frac{26^n}{26^{n+1}}\left(\frac{n}{n+1}\right)^2\right|$$

$$= \lim_{n\to\infty}\left|\frac{-5}{26}\right|\left|\left(\frac{n}{n+1}\right)^2\right| = \left|\frac{-5}{26}\right| = \frac{5}{26} < 1$$

By the ratio test, $\sum a_n$ is $\boxed{\text{ABS CONV}}$

(b)

$$\sum_{n=1}^{\infty}\left(\frac{-3n^2+n+5}{2n^2+2}\right)^n$$ root test

$$L = \lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left|\left(\frac{-3n^2+n+5}{2n^2+2}\right)^{n}\right|^{1/n} = \lim_{n\to\infty}\left|\frac{-3n^2+n+5}{2n^2+2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-3+\cancel{1/n}+\cancel{5/n^2}}{2+\cancel{2/n^2}}\right| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1 \quad \boxed{\text{DIV}} \text{ by root test}$$

2. (20 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

(a)

$$\sum(-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+7} \quad \boxed{\text{COND. CONV.}}$$

① NOT ABS. CONV.: ⓐ pick $b_n = \frac{1}{\sqrt{n}}$ so $\sum b_n$ is DIV by p-test w/ $p=\frac{1}{2}<1$

ⓑ compare $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+7} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+7}$

$= \lim_{n\to\infty} \frac{1}{1+\underbrace{7/\sqrt{n}}_{\to 0}} = 1$ nonzero const.

ⓒ By LCT, $\sum a_n$ $\boxed{\text{DIV}}$ as well.

② COND. CONV.: $\sum(-1)^n a_n$ is alt. ✓

a_n is decr. ✓

$a_n \to 0$ as $n\to\infty$ ✓

So $\sum(-1)^n a_n$ is $\boxed{\text{CONV}}$ by AST so it is $\boxed{\text{COND CONV}}$

(b)

$$\sum_{n=1}^{\infty} \left(\frac{26}{5}\right)^n \frac{1}{n!}$$

ratio test

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(\frac{26}{5}\right)^{n+1} \frac{1}{(n+1)!} \cdot \left(\frac{5}{26}\right)^n \frac{n!}{1}\right| = \lim_{n\to\infty} \left|\frac{26^{n+1}}{26^n} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{n!}{(n+1)!}\right|$

$= \lim_{n\to\infty} \left|\frac{26}{5} \frac{1}{n+1}\right| = 0 < 1$

So $\sum_{n=1}^{\infty} \left(\frac{26}{5}\right)^n \frac{1}{n!}$ is $\boxed{\text{ABS. CONV.}}$ by the ratio test.

3. (15 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(9x-2)^n}{2^n\sqrt{n}}.$$

(a) Find its radius of convergence.

ratio test $L = \lim_{n\to\infty} \left| \frac{(9x-2)^{n+1}}{2^{n+1}\sqrt{n+1}} \cdot \frac{2^n\sqrt{n}}{(9x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{9x-2}{2} \cdot \overbrace{\sqrt{\frac{n}{n+1}}}^{1} \right|$

$= \left|\frac{9x-2}{2}\right| < 1$ when $|9x-2| < 2$

or $|x - \frac{2}{9}| < \frac{2}{9}$

so $\boxed{R = \frac{2}{9}}$

(b) Find its interval of convergence.

$-\frac{2}{9} < x - \frac{2}{9} < \frac{2}{9}$

$0 < x < \frac{4}{9}$

endpts: $x=0$: $\sum_{n=1}^{\infty} \frac{(9\cdot 0-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n\cdot\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\boxed{\text{CONV}}$ by AST

$x = \frac{4}{9}$: $\sum_{n=1}^{\infty} \frac{(9\cdot\frac{4}{9}-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(4-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\boxed{\text{DIV}}$ by p-test

so the IOC is $\left[0, \frac{4}{9}\right)$

4. (15 points)

(a) Find a power series expansion of the function $f(x) = \dfrac{2x^4}{3-4x}$ about $x = 0$, write out the first five nonzero terms, and express the series in sigma notation.

$$f(x) = \frac{\frac{2}{3}x^4}{1-\frac{4}{3}x} \underset{\left|\frac{4x}{3}\right|<1}{=} \frac{2}{3}x^4 \sum_{n=0}^{\infty}\left(\frac{4}{3}x\right)^n = \sum_{n=0}^{\infty} \frac{2}{3}\cdot\frac{4^n}{3^n}\cdot x^{n+4} = \sum_{n=0}^{\infty} \frac{2\cdot 4^n}{3^{n+1}}\cdot x^{n+4}$$

$$= \frac{2}{3}x^4 + \frac{2\cdot 4}{3^2}x^5 + \frac{2\cdot 4^2}{3^3}x^6 + \frac{2\cdot 4^3}{3^4}x^7 + \frac{2\cdot 4^4}{3^5}x^8 + \cdots$$

(b) What are the radius and interval of convergence of the series you found in (a)?

By the geom. series test,

$\sum_{n=0}^{\infty} \frac{2}{3}x^4\left(\frac{4}{3}x\right)^n$ $\boxed{\text{conv}}$ for $\left|\frac{4}{3}x\right| < 1$ or $|x| < \frac{3}{4}$

and $\boxed{\text{Div}}$ for $\left|\frac{4x}{3}\right| \geq 1$

so the ROC $= \frac{3}{4}$ and the IOC $= \left(-\frac{3}{4}, \frac{3}{4}\right)$

5. (20 points)

(a) Find the Taylor series expansion of the function $f(x) = \ln(x)$ about $x = 2$, and fill in the blanks below.

$f(x) = \ln(x)$ $\qquad f(2) = \ln(2)$ $\qquad c_0 = \ln(2)$

$f'(x) = \frac{1}{x}$ $\qquad f'(2) = \frac{1}{2}$ $\qquad c_1 = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(2) = -\frac{1}{2^2}$ $\qquad c_2 = -\frac{1}{2^2 \cdot 2!}$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(2) = \frac{2}{2^3}$ $\qquad c_3 = \frac{2}{2^3 \cdot 3!}$

$f^{(4)}(x) = -\frac{2\cdot 3}{x^4}$ $\qquad f^{(4)}(2) = \frac{-2\cdot 3}{2^4}$ $\qquad c_4 = \frac{-2\cdot 3}{2^4 \cdot 4!}$

$$c_n = \frac{(-1)^{n+1}(n-1)!}{2^n \cdot n!} = \frac{(-1)^{n+1}}{2^n \cdot n}, \; n \geq 1$$

$\ln(2) + \underline{\frac{1}{2}(x-2)} + \underline{-\frac{1}{8}(x-2)^2} + \underline{\frac{1}{24}(x-2)^3} + \underline{\frac{-1}{64}(x-2)^4} + \cdots = \ln(2) + \sum_{n=1}^{\infty} \underline{\frac{(-1)^{n+1}}{2^n \cdot n}(x-2)^n}$

(b) What are the radius and interval of convergence of the series you found in (a)?

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{2^{n+1}\cdot(n+1)} \cdot \frac{2^n \cdot n}{(-1)^n (x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x-2}{2}\cdot\frac{n}{n+1}\right| = \left|\frac{x-2}{2}\right| < 1$$

$-2 < x-2 < 2$

$0 < x < 4$

endpts $x=0$: $\sum \frac{(-1)^{n+1}(-2)^n}{2^n \cdot n} = -\sum \frac{1}{n}$ $\boxed{\text{DIV}}$ (harmonic)

$x=4$: $\sum \frac{(-1)^{n+1}2^n}{2^n \cdot n} = \sum \frac{(-1)^{n+1}}{n}$ $\boxed{\text{CONV}}$ (alt. harmonic)

$\boxed{R=2}$ $\quad \boxed{\text{IOC} = (0,4]}$

6. (15 points)

(a) Find the Maclaurin series expansion of the function $f(x) = 2x\sin(2x^2)$, write out the first five nonzero terms, and express the series in sigma notation.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$2x\sin(2x^2) = 2x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x^2)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+2} \cdot x^{4n+3}$$

$$= 4x^3 - \frac{2^4}{3!} x^7 + \frac{2^8}{5!} x^{11} - \frac{2^{10}}{7!} x^{15} + \frac{2^{12}}{9!} x^{19} - \cdots$$

(b) What are the radius and interval of convergence of the series you found in (a)?

The Maclaurin series for $\sin(x)$ conv for all x

so the Maclaurin series for $f(x) = 2x\sin(2x^2)$

does as well..

$R = \infty$

$IoC = (-\infty, \infty)$