Math 143: Calculus III

Midterm I

October 6th, 2016

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 9 pages.**

QUESTION	VALUE	SCORE
1	9	
2	12	
3	10	
4	10	
5	15	
6	16	
7	14	
8	14	
TOTAL	100	

1. (9 points) Determine whether the following sequences converge. If they converge find their limit. **Justify and show all your work.**

(a)

$$a_n = \cos(n)$$

$\{a_n\}$ $\boxed{\text{DIV}}$ because $\cos(n)$ oscillates as $n \to \infty$

(b)

$$a_n = \sin\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = \boxed{0}$$

$\sin(x)$ is continuous at $x=0$

$\boxed{\text{CONV}}$

(c)

$$a_n = \ln(2n+3) - \ln(3n+2)$$

$a_n = \ln\left(\frac{2n+3}{3n+2}\right)$ by log rules.

$\ln(x)$ is continuous at $x = \frac{2}{3}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n+3}{3n+2}\right) = \ln\left(\lim_{n\to\infty} \frac{2n+3}{3n+2}\right)$$

$$= \ln\left(\lim_{n\to\infty} \frac{2+3/n}{3+2/n}\right) = \boxed{\ln\left(\frac{2}{3}\right)}$$

$\boxed{\text{CONV}}$

2. (12 points) Determine whether the following sequences converge. If they converge find their limit. **Justify and show all your work.**

(a)

$$a_n = \frac{\sin(n)}{n^2}$$

$-1 \leq \sin(n) \leq 1$

$\frac{-1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$

$\lim\limits_{n\to\infty} \frac{1}{n^2} = 0 = \lim\limits_{n\to\infty} \frac{-1}{n^2}$ so $\lim\limits_{n\to\infty} \frac{\sin(n)}{n^2} = \boxed{0}$ as well by the squeeze theorem.

$\boxed{\text{CONV}}$

(b)

$$a_n = \frac{\ln(n)}{n}$$

$\lim\limits_{n\to\infty} \frac{\ln(n)}{n} = \lim\limits_{x\to\infty} \frac{\ln(x)}{x} \underset{\substack{\uparrow \\ \text{type } \frac{\infty}{\infty} \\ \text{l'HR}}}{=} \lim\limits_{x\to\infty} \frac{1/x}{1} = \lim\limits_{x\to\infty} \frac{1}{x} = \boxed{0}$

$\boxed{\text{CONV}}$

(c)

$$a_n = \frac{e^n}{n^2+1}$$

$\lim\limits_{n\to\infty} \frac{e^n}{n^2+1} = \lim\limits_{x\to\infty} \frac{e^x}{x^2+1} \underset{\substack{\uparrow \\ \text{type } \frac{\infty}{\infty} \\ \text{l'HR}}}{=} \lim\limits_{x\to\infty} \frac{e^x}{2x} \underset{\substack{\uparrow \\ \text{type } \frac{\infty}{\infty} \\ \text{l'HR}}}{=} \lim\limits_{x\to\infty} \frac{e^x}{2} = \infty$

$\boxed{\text{DIV}}$

3. (10 points) Determine whether the following series converge or diverge and justify your answer. If they converge find their sum. **Justify and show all your work.**

(a)

$$\sum_{n=0}^{\infty} \left(\frac{-9}{5}\right)^n$$

geometric series test: $|r| = \left|-\frac{9}{5}\right| = \frac{9}{5} > 1$ $\boxed{\text{DIV}}$

(b)

$$\sum_{n=0}^{\infty} \frac{3^n + 4^n}{5^n} = \sum_{n=0}^{\infty} \left[\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n\right]$$

geometric series test: $|r_1| = \left|\frac{3}{5}\right| = \frac{3}{5} < 1$, $|r_2| = \left|\frac{4}{5}\right| = \frac{4}{5} < 1$ $\Big\}$ so both series below are $\boxed{\text{CONV}}$

$$\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{1}{1-\frac{3}{5}} = \frac{1}{2/5} = \frac{5}{2}$$

$a = 1$

$r_1 = \frac{3}{5}$

and

$$\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n = \frac{1}{1-4/5} = \frac{1}{1/5} = 5$$

$a = 1$

$r_2 = 4/5$

The sum of convergent series is convergent so

$$\sum_{n=0}^{\infty} \frac{3^n + 4^n}{5^n} = \sum_{n=0}^{\infty} \left[\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n\right] = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n = \frac{5}{2} + 5 = \boxed{7.5}$$

$\boxed{\text{CONV}}$

$\sum(a_n + b_n) = \sum a_n + \sum b_n$

4. (10 points) Determine whether the following series converge or diverge and justify your answer. If they converge find their sum. **Justify and show all your work.**

(a)

partial fractions

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \stackrel{\downarrow}{=} \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

telescoping:

$$S_k = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{k}} - \frac{1}{k+1}\right) = 1 - \frac{1}{k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \cancelto{0}{\frac{1}{k+1}}\right) = \boxed{1} \quad \boxed{\text{CONV}}$$

(b)

$$\sum_{n=1}^{\infty} e^{-n} - e^{-(n+1)}$$

telescoping:

$$S_k = \left(e^{-1} - \cancel{e^{-2}}\right) + \left(\cancel{e^{-2}} - \cancel{e^{-3}}\right) + \cdots + \left(\cancel{e^{-k}} - e^{-(k+1)}\right)$$

$$= e^{-1} - e^{-(k+1)}$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} e^{-1} - \cancelto{0}{e^{-k+1}} = e^{-1} = \frac{1}{e}$$

$$\text{so } \sum_{n=1}^{\infty} e^{-n} - e^{-(n+1)} = \boxed{\frac{1}{e}} \quad \boxed{\text{CONV}}$$

geometric series test:

$$\sum_{n=1}^{\infty}\left(e^{-n} - e^{-(n+1)}\right) = \sum_{n=1}^{\infty} e^{-n} - \sum_{n=1}^{\infty} e^{-(n+1)}$$

both series conv.

$$= \frac{e^{-1}}{1-e^{-1}} - \frac{e^{-2}}{1-e^{-1}} = \frac{e^{-1} - e^{-2}}{1 - e^{-1}} = e^{-1}\frac{\cancel{(1-e^{-1})}}{\cancel{1-e^{-1}}}$$

$a = e^{-1}$, $r = e^{-1}$; $a = e^{-2}$, $r = e^{-1}$

$$= \boxed{e^{-1}} \quad \boxed{\text{CONV}}$$

5. (15 points) Use the integral test to determine whether the following series converges or diverges. **To get full credit you must use the integral test.**

(a)

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\,\Big|_1^b = \lim_{b\to\infty}\left(\ln|b| - \cancel{\ln(1)}^{\,0}\right) = \infty \quad \boxed{\text{DIV}}$$

$\int_1^\infty \frac{1}{x}\,dx$ DIV so $\sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{DIV}}$ as well by the Integral Test since $f(x) = \frac{1}{x}$ is cont. ✓

pos. ✓

decr. ✓

(b)

$$\sum_{n=1}^{\infty} 3n^2 e^{-n^3}$$

$$\int_1^\infty 3x^2 e^{-x^3}\,dx = \lim_{b\to\infty}\int_1^b 3x^2 e^{-x^3}\,dx = \lim_{b\to\infty}\int_1^{b^3} e^{-u}\,du = \lim_{b\to\infty} -e^{-u}\Big|_1^{b^3}$$

$u = x^3$

$du = 3x^2$

$x = 1 \to u = 1$

$x = b \to u = b^3$

$$= \lim_{b\to\infty} -\cancel{e^{-b^3}}^{\,\to 0} + e^{-1}$$

$$= e^{-1} \quad \boxed{\text{CONV}}.$$

$\int_1^\infty 3x^2 e^{-x^3}\,dx$ CONV so $\sum_{n=1}^{\infty} 3n^2 e^{-n^3}$ $\boxed{\text{CONV}}$ as well by the Integral Test since $f(x) = 3x^2 e^{-x^3}$ is cont. ✓

pos. ✓

decr. ✓

6. (16 points) Use the comparison test or the limit comparison test to determine whether the following series converge or diverge. **To get full credit you must use the comparison test or the limit comparison test.**

(a)

$$\sum_{n=1}^{\infty} \frac{5^n - n}{6^n + 9} = \sum a_n \Rightarrow \text{pick } b_n = \left(\frac{5}{6}\right)^n$$

① comparison:

numerator smaller & denom. bigger

$$0 < a_n = \frac{5^n - n}{6^n + 9} < \frac{5^n}{6^n} = b_n \text{ for } n \geq 1$$

② test $\sum b_n$:

$\sum b_n$ is CONV by geom. series test w/ $|r| = \frac{5}{6} < 1$

③ CT:

$\sum a_n$ is $\boxed{\text{CONV}}$ as well by the CT.

① comparison:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5^n - n}{6^n + 9} \cdot \frac{6^n}{5^n} = \boxed{0 \text{ by l'HR}}$$

$$\lim_{n\to\infty} \frac{30^n - n6^n}{30^n + 9 \cdot 5^n} = \lim_{n\to\infty} \frac{1 - \cancel{n/5^n}}{1 + \cancel{9/6^n}} = \lim_{n\to\infty} 1 = 1 \quad \text{nonzero constant}$$

② test $\sum b_n$: $\sum b_n$ is CONV by geom. series test w/ $|r| = \frac{5}{6} < 1$

③ By LCT, $\sum a_n$ is $\boxed{\text{CONV}}$ as well

(b)

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}} = \sum a_n$$

① Comparison:

$$0 < a_n < \frac{\pi/2}{n^{1.2}} = b_n \quad \text{for } n \geq 1$$

② test b_n:

$$\sum b_n = \sum \frac{\pi/2}{n^{1.2}} = \frac{\pi}{2} \sum \frac{1}{n^{1.2}} \quad \& \quad \sum \frac{1}{n^{1.2}}$$

Const mult. of conv. series is CONV.

conv by p-test w/ $p > 1$

So $\sum b_n$ conv. as well.

③ CT:

$\sum a_n$ is $\boxed{\text{CONV}}$ as well by CT

① comparison: w/ $b_n = \frac{1}{n^{1.2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\arctan(n)}{n^{1.2}} \cdot \frac{n^{1.2}}{1} =$$

$$\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \quad \text{nonzero const}$$

② test $\sum b_n$: $\sum \frac{1}{n^{1.2}}$ CONV by p-test w/ $p = 1.2 > 1$

③ LCT:

By LCT, $\sum a_n$ $\boxed{\text{CONV}}$ as well

7. (14 points) Determine whether the following series converge or diverge. **Justify and show all your work.**

(a)

$$\sum_{n=1}^{\infty} \frac{n}{2n+5} = \sum a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+5} = \lim_{n\to\infty} \frac{1}{2+5/2} = \frac{1}{2} \neq 0$$

so $\sum a_n$ $\boxed{\text{DIV}}$ by the test for divergence.

(b)

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-\sqrt{n}} = \sum a_n$$

① comparison w/ $b_n = \frac{1}{n}$

$$a_n = \frac{\sqrt{n}}{n-\sqrt{n}} > \frac{1}{n} = b_n > 0$$

numerator larger and denominator smaller on LHS

2 test $\sum b_n$: $\sum b_n = \sum \frac{1}{n^{1/2}}$ DIV by p-test w/ $p = 1/2 < 1$

3 CT: $a_n > b_n$ & $\sum b_n$ DIV so $\sum a_n$ $\boxed{\text{DIV}}$ as well by CT.

① comparison: $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n-\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n}{n-\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{1-1/\sqrt{n}} = 1$$ nonzero const.

② test $\sum b_n$: $\sum \frac{1}{n^{1/2}}$ DIV by p-test w/ $p = 1/2 < 1$

③ LCT

$\sum a_n$ $\boxed{\text{DIV}}$ as well by LCT

8. (14 points) Determine whether the following series converge or diverge. **Justify and show all your work.**

(a)

$$\sum_{n=1}^{\infty} \frac{n}{n^3+5n} = \sum a_n \qquad \underline{\text{LCT}}$$

① choose b_n: $b_n = \frac{n}{n^3} = \frac{1}{n^2}$

② test $\sum b_n$: $\sum b_n = \sum \frac{1}{n^2}$ $\boxed{\text{CONV}}$ by p-test w/ $p=2>1$

③ LCT: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3+5n} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^3}{n^3+5n} = \lim_{n\to\infty} \frac{1}{1+5/n^2}$

$= 1$ nonzero const.

By the LCT, $\sum a_n$ $\boxed{\text{conv.}}$ as well.

(b)

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} = \sum a_n \qquad \text{CT}$$

① choose $b_n = \frac{1}{n}$

② test $\sum b_n$: $\sum b_n = \sum \frac{1}{n}$ DIV (harmonic series)

comparison:

③ $\ln(n) \geq 1$ for $n \geq 3$

So $\frac{\ln(n)}{n} \geq \frac{1}{n} > 0$ for $n \geq 3$

④ Thus, by the CT, $\sum a_n$ $\boxed{\text{DIV}}$ as well.