Math 143: Calculus III

Midterm 1

February 17, 2015

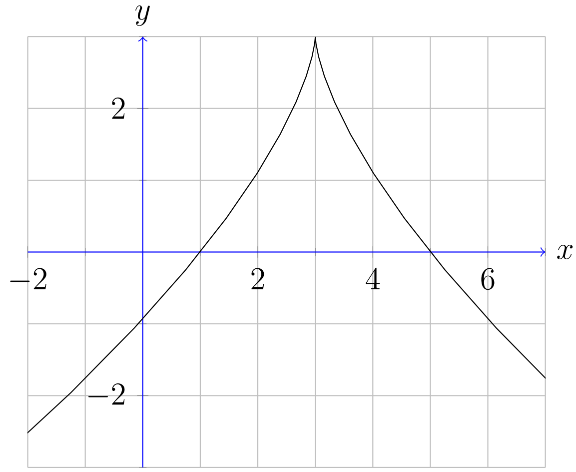
NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
- **No notes, papers, books, or other aids are permitted. A formula sheet appears at the end of the exam; feel free to tear it out.**
- **Show work and justify all answers.** Box **final answers.**
- **You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Numerical or algebraic simplifications of answers are not required, EXCEPT when specifically stated otherwise.**
- **You are responsible for checking that this exam has all 10 pages.**

QUESTION	VALUE	SCORE
1	18	
2	8	
3	6	
4	12	
5	12	
6	12	
7	7	
8	9	
9	8	
10	8	
TOTAL	100	

1. (18 points) Consider the parametric curve $x = 2t^3 + 3$, $y = 3 - 3t^2$, whose graph appears below:



(a) Find all points on the curve such that the slope of the tangent line is $\frac{1}{2}$.

(b) Find the coordinates of the two intersection points of the curve with the x-axis, and the corresponding values of t.

(c) Find the area of the region lying below the curve and above the x-axis.

2. (8 points) Make the following coordinate conversions. (Your answers should not have any functions in them.)

(a) Convert the point $(x, y) = (-1, \sqrt{3})$ to polar coordinates.

(b) Convert the point $(r, \theta) = (\sqrt{3}, \pi/6)$ to rectangular coordinates.

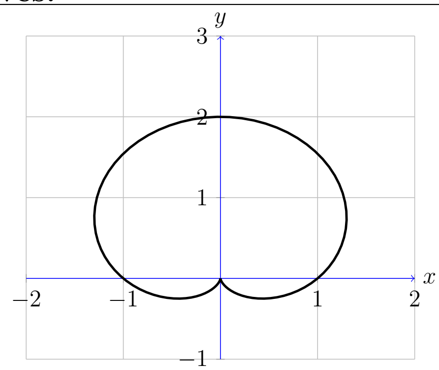
(c) Convert the point $(x, y) = (6, -6)$ to polar coordinates.

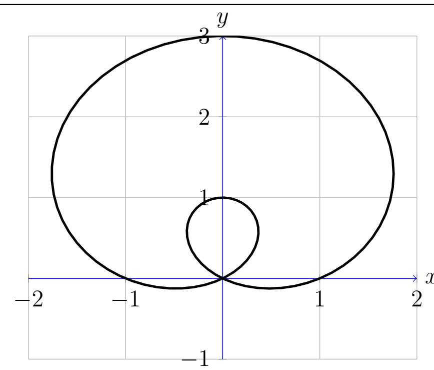
(d) Convert the point $(r, \theta) = (-6, -5\pi)$ to rectangular coordinates.

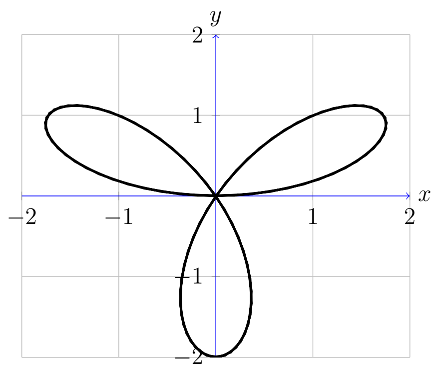
3. (6 points) Sketch the polar region defined by the inequalities $-\frac{\pi}{4} \leq \theta \leq \pi$, $1 \leq r \leq 3$. Make sure to label any relevant distances on the x and y axes.

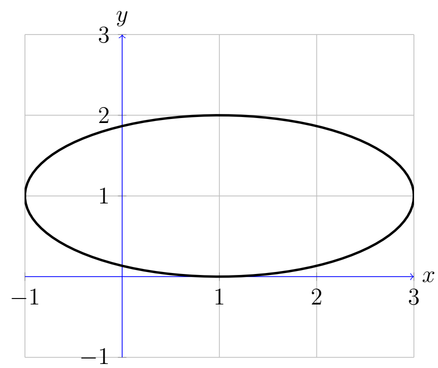
4. (12 points) Match the following curves to their equations.

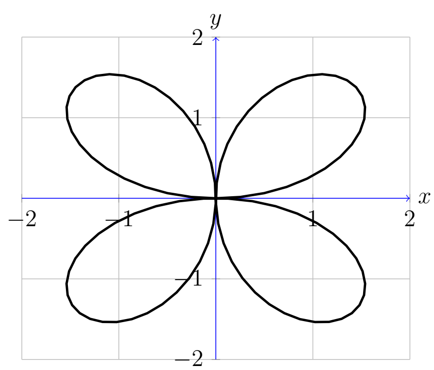
Curves:

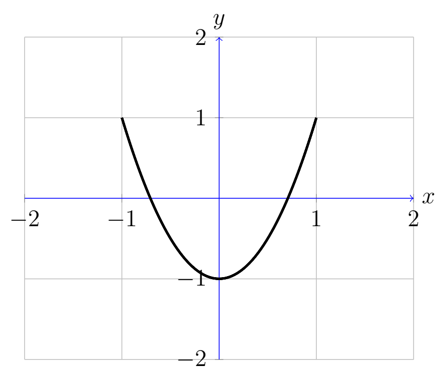
(a)


(b)


(c)


(d)


(e)


(f)


Equations:

1.) $x = \sin(t)$,
$y = 2\sin^2(t) - 1$

2.) $r = 2\sin(2\theta)$

3.) $r = 1 + 2\cos(\theta)$

4.) $x = \cos^2(t)$,
$y = 1 - 2\sin(t)$

5.) $r = 2\sin(3\theta)$

6.) $r = 1 + 2\sin(\theta)$

7.) $x = 1 + 2\cos(t)$,
$y = 1 + \sin(t)$

8.) $r = 2\sin(4\theta)$

9.) $r = 1 + \sin(\theta)$

Answers:

(a)________ (b)________ (c)________ (d)________ (e)________ (f)________

5. (12 points) Consider the polar curve $r = 1 + \sin(6\theta)$ for $0 \leq \theta \leq 2\pi$.

(a) Find all values of θ in the interval $[0, 2\pi]$ where this curve passes through the origin.

(b) Sketch the graph of this curve. Make sure to label any relevant distances on the x and y axes.

6. (12 points) At time t seconds ($t \geq 0$), a particle's position in the plane is given by

$$x = e^t + e^{-t}, \quad y = 2t + 5.$$

(a) Is the particle ever moving directly in the vertical direction? Directly in the horizontal direction? If so, when?

(b) Find the distance traveled by the particle between time $t = 0$s and time $t = 1$s.

7. (7 points) Find the area enclosed by the polar curve $r = \sqrt{3 + 2\sin(4\theta)}$.

8. (9 points) Let $z = 2 - 2i$ and $w = 5 + i$. Compute:

(a) $z^2 + 2w$, in $a + bi$ form.

(b) $\dfrac{z}{w}$, in $a + bi$ form.

(c) Real numbers r and θ such that $z = r\,e^{i\theta}$.

9. (8 points) Find, in $a + bi$ form, all complex numbers z such that $z^4 = -16$.

10. (8 points) Two complex numbers have a sum of 4 and a product of 6. Find them.

Common Formulas

- Trigonometry:
  - Definitions: $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$, $\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
$\tan(\theta)$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undef.	$-\sqrt{3}$	-1	$-1/\sqrt{3}$	0

  - Addition formula for sine: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
  - Addition formula for cosine: $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.
  - Power-reduction formulas: $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$.
  - Pythagorean identity: $\sin^2(\theta) + \cos^2(\theta) = 1$.

- For the parametric curve $x = x(t)$, $y = y(t)$ on the interval $t_1 \le t \le t_2$:
  - Tangent slope $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{y'(t)}{x'(t)}$.
  - Arclength $s = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$.
  - Area $A = \begin{cases} -\int_{t_1}^{t_2} y(t)\,x'(t)\,dt & \text{for a curve above the } x\text{-axis traced left to right} \\ \int_{t_1}^{t_2} y(t)\,x'(t)\,dt & \text{for a curve above the } x\text{-axis traced right to left} \end{cases}$
  - Area $A = \int_{t_1}^{t_2} x(t)\,y'(t)\,dt = \frac{1}{2}\int_{t_1}^{t_2} [x(t)y'(t) - x'(t)y(t)]\,dt$ for a closed curve traced counterclockwise.

- Polar (r,θ) and rectangular (x,y) coordinate conversions:
  - Cartesian to Polar: $x = r\cos(\theta)$, $y = r\sin(\theta)$.
  - Polar to Cartesian: $r = \sqrt{x^2+y^2}$, $\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \pi + \tan^{-1}(y/x) & \text{if } x < 0 \end{cases}$.

- For the polar curve $r = r(\theta)$ on the interval $\theta_1 \leq \theta \leq \theta_2$:
  - Tangent slope $\dfrac{dy}{dx} = \dfrac{r' \sin(\theta) + r \cos(\theta)}{r' \cos(\theta) - r \sin(\theta)}$.
  - Arclength $s = \displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2}\, d\theta$.
  - Area of "polar sector" $A = \begin{cases} \dfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta & \text{for a curve traced counterclockwise} \\ -\dfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta & \text{for a curve traced clockwise} \end{cases}$

- If $z = x + yi$ is a complex number, where $i^2 = -1$:
  - The complex conjugate is $\overline{z} = x - yi$.
  - The magnitude (or length) is$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$.
  - The argument (or angle) is $\arg(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \pi + \tan^{-1}(y/x) & \text{if } x < 0 \end{cases}$.
- Euler's identity: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.
- We can write $z = r\,[\cos(\theta) + i \sin(\theta)] = r\, e^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$.
- The n solutions to $z^n = r\, e^{i\theta}$ are $z = r^{1/n} e^{i\theta/n} e^{2\pi ik/n}$ where $k = 0, 1, \cdots, n-1$.
- The two solutions to $az^2 + bz + c = 0$ are $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
