| 143 s08 | Midterm 1 | Exam Time: R 2/21, 8:00 - 9:30 |
|---|---|---|
| Name: | Section: | Student No.: |

**Instructions:**

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT seperate the pages of your exam.

| Problem | Points | Score |
|---|---|---|
| A1 | 12 | |
| A2 | 10 | |
| A3 | 12 | |
| A4 | 8 | |
| A5 | 8 | |
| Total | 50 | |

Name:

**Section A:** Answer ALL questions.

**Problem A1:** [12 pts] Do the following sequences converge or diverge? If they converge, give the limit. If they diverge, specify whether the limit is infinity ($\infty$), negative infinity ($-\infty$) or does not exist (DNE) Justify your answers.

(**a**) $a_n = \dfrac{\ln(\ln n)}{\ln n}$

**Solution:**

Set $f(x) = \frac{\ln(\ln x)}{\ln x}$ then $b_n = f(n)$ and so

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\ln x} = 0$$

$$\lim_{n\to\infty} a_n = \quad 0$$

(**b**) $b_n = \dfrac{3^n \cos n}{4^n}$

**Solution:**

Note $-1 \le \cos n \le 1$ so

$$-\left(\frac{3}{4}\right)^n \le c_n \le \left(\frac{3}{4}\right)^n$$

The sequence on the left and the right are both geometric with ratio $3/4$, so both converge to 0. By the squeeze theorem, $c_n \to 0$.

$$\lim_{n\to\infty} b_n = \quad 0$$

(**c**) $c_n = \left(\dfrac{2n^3 + n - 1}{n^3 - n^2 + 1}\right)^2$

**Solution:**

Divide the inside top and bottom by $n^2$, then

$$a_n = \left(\frac{2 + 1/n^2 - 1/n^3}{1 - 1/n + 1/n^3}\right)^2 \to \left(\frac{2 + 0 - 0}{1 - 0 + 0}\right)^2 = 4$$

$$\lim_{n\to\infty} c_n = \quad 4$$

Name:

**Problem A2:** [10 pts] Evaluate the sums of the following series exactly. If the series happens to diverge give your answer as $\infty$, $-\infty$ or $DNE$ as appropriate. Justify your answers.

**(a)** $\sum_{n=0}^{\infty} (\arctan(n+1) - \arctan(n))$

**Solution:**

If we write out the $n$th partial sum, we see

$$\begin{aligned} s_n = &(\arctan(1) - 0) + (\arctan(2) - \arctan(1)) + (\arctan(3) - \arctan(2)) \\ &+ \cdots + (\arctan(n) - \arctan(n-1)) + (\arctan(n+1) - \arctan(n)). \end{aligned}$$

Most terms cancel out and we see $s_n = \arctan(n+1) - 0$. The sequence of partial sums converges to $\frac{\pi}{2}$ and so the series converges to $\frac{\pi}{2}$ also.

$\frac{\pi}{2}$

**(b)** $\sum_{n=3}^{\infty} \frac{2}{3^{2n-4}}$

**Solution:**

The sequence is geometric with ratio $r = \frac{1}{3^2} = \frac{1}{9}$ and first term $a = 2(3)^{-2} = \frac{2}{9}$. The series therefore converges to $\frac{2/9}{1-1/9} = \frac{1}{4}$.

$\frac{1}{4}$

Name:

**Problem A3:** [12 pts] Do the following series converge (C) or diverge (D)? Justify your answers, explicitly stating any tests or theorems that you use.

(**a**) $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{2\sqrt{n}}$

**Solution:**

Set $f(x) = \frac{e^{-\sqrt{x}}}{2\sqrt{x}}$. Then for $x > 1$, we have that $f(x) > 0$. Since the denominator terms are both increasing, we see that $f(x)$ is decreasing and so the integral test applies. Now using the substitution $u = \sqrt{x}$ so $du = \frac{1}{2\sqrt{x}}dx$ we see

$$\int_{x=1}^{\infty} \frac{e^{-\sqrt{x}}}{2\sqrt{x}} dx = \int_{u=1}^{\infty} e^{-u} du = -e^{-u}\Big|_1^{\infty} = \frac{1}{e} < \infty$$

Thus by the integral test the series converges.

| |
|---|
| $C$ |

(**b**) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n!}\right)$

**Solution:**

Set $a_n = \cos\left(\frac{1}{n!}\right)$, then

$$\lim_{n\to\infty} a_n = \cos(0) = 1 \neq 0$$

Thus by the divergence test, $\sum a_n$ must diverge.

| |
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| $D$ |

Name:

**(c)** $\sum_{n=1}^{\infty} \frac{n^2+n+1}{3n^4-n-2}$

**Solution:**

The largest term on top is $n^2$, the largest term on the bottom is $n^4$ so we shall limit compare to $\sum \frac{n^2}{n^4}$. Now the quotient becomes

$$\frac{n^2+n+1}{n^2}\frac{n^4}{3n^4-n-2} = \frac{1+1/n+1/n^2}{3-1/n^3-2/n^4} \to \frac{1}{3}.$$

The comparison is therefore valid. Now $\sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$ which is a $p$-series with $p = 2 > 1$ so converges. By the limit comparison test, the series above converges also.

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|---|
| $C$ |

Name:

**Problem A4:** [8 pts] Does the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2}$$

converge or diverge? If it converges, estimate the sum accurate to within 1/100. If it diverges, state whether it diverges to $\infty$, $-\infty$ or neither. Justify your solution.

**Solution:**

The sequence $b_n = \frac{1}{(n!)^2}$ is positive, decreasing and $b_n \to 0$. Thus the alternating series test applies and shows the alternating series $\sum(-1)^n b_n$ converges. To estimate the series to within 1/100, we must use $s_n$ where $n$ is chosen to make $b_{n+1} < 1/100$.

Write out the first few terms of the sequence $b_0 = 1$, $b_1 = 1$, $b_2 = 1/4$, $b_3 = 1/36$, $b_4 = 1/(24)^2$ ...

Thus $b_4 < 1/100$ and so

$$s_3 = 1 - 1 + \frac{1}{4} - \frac{1}{36} = \frac{9-1}{36} = \frac{2}{9}$$

is an approximation to the sum of the series accurate to within 1/100.

$\frac{2}{9}$

Name:

**Problem A5:** [8 pts] The series $\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$ is approximated by its 3rd partial sum

$$s_3 = 0 + \frac{2}{4} + \frac{4}{25} + \frac{6}{100}.$$

How large can the error in this approximation be? (If the series diverges, your answer will be $\infty$)

**Solution:**

The function $f(x) = \frac{2x}{(x^2+1)^2}$ is positive for $x > 0$. Now

$$f'(x) = \frac{2(x^2+1)^2 - 4x^2(x^2+1)}{(x^2+1)^4} = \frac{2-2x^2}{(x^2+1)^3}$$

which is negative for $x > 1$. Therefore the integral series test applies and the estimation theorem states that

$$R_3 \leq \int_3^{\infty} f(x)dx = \int_3^{\infty} \frac{2x}{(x^2+1)^2} dx$$

Using $u = x^2 + 1$ so $du = 2xdx$, we see

$$R_3 \leq \int_{10}^{\infty} \frac{1}{u^2} du = -\frac{1}{u}\Big|_{10}^{\infty} = \frac{1}{10}$$

max error = $\frac{1}{10}$