Math 143: Calculus III

Midterm 2 Solutions

1. (10 points)

(a) Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{3}{2^{n+1}\sqrt{n+1}}(x+2)^{n+1}}{\frac{3}{2^n\sqrt{n}}(x+2)^n}\right| = \lim_{n\to\infty}\left|\frac{3(x+2)^{n+1}}{2^{n+1}\sqrt{n+1}}\frac{2^n\sqrt{n}}{3(x+2)^n}\right| = \frac{|x+2|}{2}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x+2|}{2}$$

We know the series is absolutely convergent when $|x+2| < 2$ which means the radius of convergence is $R = 2$.

(b) $|x+2| = 2$ gives endpoints $x = 0$ and $x = -4$.

When $x = 0$, the series is $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3}{2^n\sqrt{n}}2^n = \sum_{n=1}^{\infty}(-1)^n\frac{3}{\sqrt{n}}$. This is a decreasing alternating series with $\displaystyle\lim_{n\to\infty}\frac{3}{\sqrt{n}} = 0$ and so it converges by the Alternating Series Test.

When $x = -4$, the series is $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3}{2^n\sqrt{n}}(-2)^n = \sum_{n=1}^{\infty}\frac{3}{\sqrt{n}}$. This is a p-series with $p = 1/2 < 1$, hence it diverges.

The interval of convergence is $(-4, 0]$.

2. (10 points)

(a) Using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)}{4!}(x-1)^{n+1}}{\frac{n}{4!}(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-1)^{n+1}}{4!}\frac{4!}{n(x-1)^n}\right| = |x-1|\lim_{n\to\infty}\frac{n+1}{n} = |x-1|$$

We know the series is absolutely convergent when $|x-1| < 1$ which means the radius of convergence is $R = 1$.

(b) $|x-1| = 1$ gives endpoints $x = 0$ and $x = 2$.

When $x = 0$, the series is $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{4!}$. This is an alternating series, BUT $\displaystyle\lim_{n\to\infty}(-1)^n\frac{n}{4!} \neq 0$ and so the series diverges by the Divergence Test.

When $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{n}{4!}$. Since $\lim_{n\to\infty} \frac{n}{4!} = \infty \neq 0$, this series also diverges by the Divergence Test.

The interval of convergence is $(0, 2)$.

3. (15 points)

(a) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ has radius of convergence 1.

(b) $\frac{1}{4-x} = \frac{1}{4(1-x/4)} = \frac{1}{4} \cdot \frac{1}{1-x/4} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty} 4^{-n-1} x^n$. For convergence we need $|x/4| < 1$, so $|x| < 4$, which gives a radius of convergence of 4.

(c) $\frac{1}{4-x} = \frac{1}{1-(x-3)} = \sum_{n=0}^{\infty} (x-3)^n$. For convergence, we need $|x-3| < 1$, so the radius of convergence is 1.

4. (15 points)

(a) Replace x by $3x^2$ in the given formula, and then multiply by 5:

$$5 \arctan(3x^2) = 5 \sum_{n=0}^{\infty} (-1)^n \frac{(3x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{5 \cdot 3^{2n+1}}{2n+1} x^{4n+2}.$$

This change requires $|3x^2| < 1$ and the radius of convergence is $1/\sqrt{3}$.

(b) We note that

$$\frac{1}{1+x^2} = \frac{d}{dx} \arctan x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Taking derivatives does not change the radius of convergence and since $\arctan x$ has radius of convergence 1, so will $\frac{1}{1+x^2}$.

(c) We replace x in (b) by $4x^2$ and then multiply by $3x$:

$$\frac{3x}{1+(4x^2)^2} = 3x \sum_{n=0}^{\infty} (-1)^n (4x^2)^{2n} = \sum_{n=0}^{\infty} (-1)^n 3 \cdot 16^n x^{4n+1}$$

This substitution requires $|4x^2| < 1$ and so the radius of convergence for this series is $1/2$.

5. (20 points)

(a)

$$\begin{array}{lll}
f(x) = \sin x & f(\pi/2) = 1 & c_0 = 1 \\
f'(x) = \cos x & f'(\pi/2) = 0 & c_1 = 0 \\
f''(x) = -\sin x & f''(\pi/2) = -1 & c_2 = -1/2! \\
f^{(3)}(x) = -\cos x & f^{(3)}(\pi/2) = 0 & c_3 = 0 \\
f^{(4)}(x) = \sin x & f^{(4)}(\pi/2) = 1 & c_4 = 1/4! \\
f^{(5)}(x) = \cos x & f^{(5)}(\pi/2) = 0 & c_5 = 0 \\
f^{(6)}(x) = -\sin x & f^{(6)}(\pi/2) = -1 & c_6 = -1/6!
\end{array}$$

(b) $T_4(x) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4$

(c) $T_4\left(\frac{7\pi}{12}\right) = 1 - \frac{1}{2!}\left(\frac{7\pi}{12} - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(\frac{7\pi}{12} - \frac{\pi}{2}\right)^4 = 1 - \frac{1}{2}\left(\frac{\pi}{12}\right)^2 + \frac{1}{24}\left(\frac{\pi}{12}\right)^4$

(d) Using the Alternating Series Estimate, we know the remainder $|R_4(x)|$ is less than the next term in the series, namely, $\left|\frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6\right|$. Thus,

$$\left|R_4\left(\frac{7\pi}{12}\right)\right| \leq \left|\frac{1}{6!}\left(\frac{7\pi}{12} - \frac{\pi}{2}\right)^6\right| = \frac{1}{6!}\left(\frac{\pi}{12}\right)^6.$$

6. (15 points)

(a)

$$\begin{array}{ll}
f(x) = e^x & f(0) = 1 \\
f'(x) = e^x & f'(0) = 1 \\
f''(x) = e^x & f''(0) = 1 \\
f^{(n)}(x) = e^x & f^{(n)}(0) = 1
\end{array}$$

Therefore,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(b) Replacing x by $(-x^2)$ in the expression above,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!},$$

$$\int e^{-x^2}\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}$$

(c)

$$\int_0^1 e^{-x^2}\,dx \approx \left[x - \frac{x^3}{3\cdot 1!}\right]_0^1 = \frac{2}{3}$$

7. (10 points) We know (or can easily compute) that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The 3rd Taylor polynomial for $\cos x$ is then

$$T_3(x) = 1 - \frac{x^2}{2!},$$

and the remainder is bounded by the next term in the series (using the Alternating Series Estimate), namely $\left|\frac{x^4}{4!}\right|$. Since we need $|R_3(x)| \le 10^{-2}$, it will be satisfied by those x for which

$$\begin{aligned}\left|\frac{x^4}{4!}\right| &\le 10^{-2}\\ x^4 &\le .24\\ |x| &\le \sqrt[4]{.24}\end{aligned}$$

That is, the estimate is accurate within 10^{-2} whenever $x \in [-\sqrt[4]{.24}, \sqrt[4]{.24}]$.

8. (5 points) Recall that a Taylor polynomial has coefficients $c_n = \frac{f^{(n)}(a)}{n!}$. In this example, $a = 3$ and then $f^{(n)}(3) = c_n \cdot n!$

$$\begin{aligned}f(3) &= c_0 \cdot 0! = 1 \cdot 1 = 1\\ f''(3) &= c_2 \cdot 2! = 0 \cdot 2! = 0\\ f^{(100)}(3) &= c_{100} \cdot 100! = -99 \cdot 100!\end{aligned}$$