Math 143 - Fall 2008

Solutions to Midterm 1

**1. (20 points)**

(a) $\lim_{n\to\infty} 4\arctan(n) = 4\cdot\frac{\pi}{2} = 2\pi$

(b) $\lim_{n\to\infty} \sin(n) = DNE$ because as $n$ gets large, $\sin(n)$ oscillates between 1 and -1.

(c) $\lim_{n\to\infty} \frac{2n+1}{n^2+3} = \lim_{n\to\infty} \frac{2n/n^2+1/n^2}{1+3/n^2} = 0.$

(d) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{3n} = "1^{\infty}".$

We apply ln and then use L'Hospital's Rule:

$$\begin{aligned}
\lim_{n\to\infty} 3n\ln\left(1+\frac{2}{n}\right) &= 3\lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}\\
&= 3\lim_{n\to\infty}\frac{\frac{1}{1+\frac{2}{n}}\left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}}\\
&= 3\lim_{n\to\infty}\frac{2}{1+\frac{2}{n}}\\
&= 6
\end{aligned}$$

Thus, $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{3n} = e^6.$

**2. (10 points)**

$$\begin{aligned}
8.626262\ldots &= 8+\frac{6}{10}+\frac{2}{100}+\frac{6}{10^3}+\frac{2}{10^4}+\cdots\\
&= 8+\frac{62}{100}+\frac{62}{10^4}+\cdots\\
&= 8+\frac{62/100}{1-1/100}\\
&= 8+\frac{62}{100}\frac{100}{99}\\
&= 8+\frac{62}{99}.
\end{aligned}$$

The third step comes from the fact that after 8, the sum is a geometric series with $a = 62/100$ and $r = 1/100$.

**3. (10 points)** First note that

$$\frac{2}{(n+1)(n+2)} = \frac{2}{n+1} - \frac{2}{n+2}$$

Then the $n^{th}$ partial sum has the formula:

$$s_n = \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \cdots + \left(\frac{2}{n+1} - \frac{2}{n+2}\right) = 1 - \frac{2}{n+2}.$$

Taking the limit as $n \to \infty$, we get

$$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{2}{n+2}\right) = 1.$$

**4. (20 points)**

(a) 1. $\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$

2. Since $n+1 > n$ and $\ln(n+1) > \ln(n)$, we have $(n+1)\ln(n+1) > n\ln(n)$ and it follows that $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln(n)}$.

The test implies (circle ONE): **convergence** divergence inconclusive.

(b)

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\frac{1}{(n+1)\ln(n+1)}}{(-1)^n\frac{1}{n\ln(n)}} \right| &= \lim_{n\to\infty} \frac{n}{n+1}\frac{\ln(n)}{\ln(n+1)} \\ &= \lim_{n\to\infty} \frac{n}{n+1} \lim_{n\to\infty} \frac{1/n}{1/(n+1)} \\ &= 1. \end{aligned}$$

The test implies (circle ONE): absolute convergence divergence **inconclusive**.

(c) (i)

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{n\ln(n)}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{n^2}{n\ln(n)} \\ &= \lim_{n\to\infty} \frac{n}{\ln(n)} \\ &= \lim_{n\to\infty} \frac{1}{1/n} \\ &= \lim_{n\to\infty} n \\ &= \infty \end{aligned}$$

The test implies (circle ONE): convergence divergence **inconclusive**.

(ii) Using the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ we have

$$\int_2^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln 2}^{\infty} \frac{du}{u} = \ln(u)\Big|_{\ln 2}^{\infty} = \infty.$$

The test implies (circle ONE): convergence **divergence** inconclusive.

Then based on the four tests above, the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n\ln(n)}$ is (circle ONE):

absolutely convergent **conditionally convergent** divergent unknown.

**5. (15 points)**

(a) Using the integral test and substitution $u = \ln(x)$, $du = \frac{dx}{x}$ we get

$$\int_2^\infty \frac{dx}{x(\ln(x))^2} = \int_{\ln 2}^\infty \frac{du}{u^2} = -\frac{1}{u}\Big|_{\ln 2}^\infty = 0 - \left(-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Hence, by Integral Test, the series converges.

(b) Using the Ratio Test,

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{3^{n+1}\ln(n+1)}{7\,(n+1)!}}{\frac{3^n\ln(n)}{7\,n!}} &= \lim_{n\to\infty} \frac{3^{n+1}\ln(n+1)}{7\,(n+1)!}\frac{7\,n!}{3^n\ln(n)} \\ &= \lim_{n\to\infty} \frac{3\ln(n+1)}{(n+1)\ln(n)} \\ &= 0.\end{aligned}$$

By the Ratio Test, since $0 < 1$, the series converges absolutely.

(c)

$$\lim_{n\to\infty} \frac{2n}{\sqrt{n^2+3}} = \lim_{n\to\infty} \frac{2}{\sqrt{1+3/n^2}} = 2.$$

Since this limit is not zero, the series diverges by the Divergence Test.

**6. (10 points)**

(a) To check for absolute convergence, let us look at $\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$. This is $p$-series with $p = 2 > 1$, hence it converges. Therefore, the alternating series above is absolutely convergent.

(b) Since the series is alternating, we know that

$$|R_n| \le \frac{1}{(n+1)^2}.$$

We need this remainder to be less than or equal to $10^{-4}$. Thus,

$$\begin{aligned}\frac{1}{(n+1)^2} &\le \frac{1}{10^4} \\ (n+1)^2 &\ge 10^4 \\ n+1 &\ge 100 \\ n &\ge 99.\end{aligned}$$

The smallest $n$ we can choose is $n = 99$.

**7. (9 points)** Using the Ratio Test, we have

$$\lim_{n\to\infty}\left|\frac{\frac{2\pi(x-1)^{n+1}}{n+1}}{\frac{2\pi(x-1)^n}{n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x-1| = |x-1|.$$

When $|x-1|<1$, the power series is absolutely convergent and it diverges for $|x-1|>1$.
To check at endpoints, we solve for $|x-1|=1$ and get $x=0$ and $x=2$.
At $x=0$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{2\pi}{n}(0-1)^n = 2\pi\sum_{n=1}^{\infty}\frac{1}{n}$, which is divergent (harmonic series).
At $x=2$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{2\pi}{n}(2-1)^n = 2\pi\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, which is conditionally convergent(alternating harmonic series).
Therefore, the interval of convergence for this power series is $(0,2]$.

**8. (6 points)**

(a) $\lim_{n\to\infty} a_n = 0$. That is because the series converges to 2, and the only way a series converges is if the terms $a_n$ get small (approach zero).

(b) $\lim_{n\to\infty} a_{n+2} = 7$. The limit just means that the sequence $a_n$ approaches 7. Of course, that also means $a_{n+2}$ approaches 7.