1. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use and justify its use.*

1.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^{3/2}}{n^2-6} = \sum a_n$$

abs. conv.? $|a_n| = \dfrac{n^{3/2}}{n^2-6}$ for $n \geq 3$

① pick $b_n = \dfrac{n^{3/2}}{n^2} = \dfrac{1}{n^{1/2}}$ so $\sum b_n$ DIV by p-test

② $\lim\limits_{n\to\infty}\dfrac{|a_n|}{b_n} = \lim\limits_{n\to\infty}\dfrac{n^{3/2}}{n^2-6}\cdot\dfrac{n^{1/2}}{1} = \lim\limits_{n\to\infty}\dfrac{n^2}{n^2-6} = \lim\limits_{n\to\infty}\dfrac{1}{1-6/n^2} = 1$

non-zero const.

③ So, by LCT, $\sum |a_n|$ DIV as well, so not abs. conv.

cond. conv.? AST: alt. ✓ (for $n \geq 3$)

decr. ✓ $f'(x) = \dfrac{3/2\, x^{1/2}\cdot(x^2-6) - x^{3/2}\cdot 2x}{(x^2-6)^2} = \dfrac{3/2\, x^{5/2} - 9x^{1/2} - 2x^{5/2}}{(x^2-6)^2} = \dfrac{-9x^{1/2} - 1/2\, x^{5/2}}{(x^2-6)^2} < 0$ for all $x > 0$

$\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\dfrac{n^{3/2}}{n^2-6} = \lim\limits_{n\to\infty}\dfrac{1}{n^{1/2} - 6/n^{3/2}} = 0$ ✓

$\sum a_n$ is $\boxed{\text{COND. CONV.}}$

2.

$$\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2} = \sum a_n$$

abs. conv.? $|a_n| = \left|\dfrac{\sin(n)}{n^2}\right| \leq \dfrac{1}{n^2}$

① pick $b_n = \dfrac{1}{n^2}$ so $\sum b_n$ CONV. by p-test

② By CT, $\sum |a_n|$ conv. as well

③ So $\sum a_n$ is $\boxed{\text{ABS. CONV.}}$

2. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use and justify its use.*

1.

$$\sum_{n=1}^{\infty}(-1)^n\frac{3\ln(n)}{n} = \sum a_n$$

ABS CONV.? $|a_n| = \frac{3\ln(n)}{n} > \frac{1}{n}$

① pick $b_n = \frac{1}{n}$ so $\sum b_n$ DV by p-test

② Then, $\sum|a_n|$ DV as well, by CT since $|a_n| > b_n$

③ so $\sum a_n$ is not abs. conv.

COND. CONV.?

AST: alt. ✓ for $n \geq 2$

decr. ✓ $f'(x) = \frac{(3/x)\cdot x - 3\ln(x)\cdot 1}{x^2} = \frac{3 - 3\ln(x)}{x^2} < 0$ for $x \geq$

$\lim_{n\to\infty} \frac{3\ln(n)}{n} \underset{\uparrow\ \frac{\infty}{\infty}\text{ L'HR}}{=} \lim_{n\to\infty}\frac{3/n}{1} = 0$ ✓

$\boxed{\text{So } \sum a_n \text{ is COND. CONV.}}$

2.

$$\sum a_n = \sum_{n=1}^{\infty}(-1)^n\frac{3\ln(n)}{n^2}$$

ABS. CONV.? $|a_n| = \frac{3\ln(n)}{n^2} = \left(\frac{\ln(n)}{n^{.5}}\right)\cdot\left(\frac{3}{n^{1.5}}\right) < \frac{3}{n^{1.5}}$ for all n since $\ln(n) < \sqrt{n}$

① pick $b_n = \frac{3}{n^{1.5}}$ so $\sum b_n = 3\sum\frac{1}{n^{1.5}}$ CONV. by p-test (and const. • conv. = conv.)

② Then $\sum|a_n|$ by CT conv. as well since $|a_n| < b_n$ (since $\frac{\ln(n)}{n^{.5}} < 1$ for all n)

③ So $\sum a_n$ is $\boxed{\text{abs. conv.}}$

3. (20 points) Find the radius and interval of convergence of the following power series.

$$\sum_{n=3}^{\infty} \frac{(-5)^n(x-3)^n}{(n-2)^{3/2}4^n}.$$

<u>Ratio test</u>: $L = \lim_{n\to\infty} \left| \frac{(-5)^{n+1}(x-3)^{n+1}}{(n-1)^{3/2}\,4^{n+1}} \cdot \frac{(n-2)^{3/2}\,4^n}{(-5)^n(x-3)^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{5}{4}(x-3) \cdot \cancel{\left(\frac{n-2}{n-1}\right)^{3/2}} \right| = \frac{5}{4}|x-3|$$

1 as $n \to \infty$

so $L < 1$ when $\frac{5}{4}|x-3| < 1$ or $|x-3| < \frac{4}{5}$

$$-\frac{4}{5} < x-3 < \frac{4}{5}$$

$$3 - \frac{4}{5} < x < \frac{4}{5} + 3$$

$$\frac{11}{5} < x < \frac{19}{5}$$

<u>endpts.</u> $x = \frac{11}{5}$: $\sum_{n=3}^{\infty} \frac{(-5)^n\left(-\frac{4}{5}\right)^n}{(n-2)^{3/2}4^n} = \sum_{n=3}^{\infty} \frac{1}{(n-2)^{3/2}}$ <u>CONV</u> by LCT w/ $\frac{1}{n^{3/2}}$ & p-test

$x = \frac{19}{5}$: $\sum_{n=3}^{\infty} \frac{(-5)^n\left(\frac{4}{5}\right)^n}{(n-2)^{3/2}4^n} = \sum_{n=3}^{\infty} \frac{(-1)^n}{(n-2)^{3/2}}$ <u>CONV.</u> by AST

$\boxed{R = \frac{4}{5}}$ IOC $= \left[\frac{11}{5}, \frac{19}{5}\right]$

4. (20 points)

(a) Consider the function $f(x) = \ln(2x)$. Find a power series expansion of $f(x)$ about $x = 3$.

$f(x) = \ln(2x)$

$f'(x) = \frac{1}{2x}\cdot 2 = \frac{1}{x} = x^{-1}$

$f''(x) = -\frac{1}{x^2} = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(4)}(x) = 2\cdot(-3)\cdot x^{-4}$

$f(3) = \ln(6)$

$f'(3) = \frac{1}{3}$

$f''(3) = -\frac{1}{3^2}$

$f'''(3) = 2\cdot\frac{1}{3^3} = \frac{2!}{3^3}$

$f^{(4)}(3) = -2\cdot 3\cdot\frac{1}{3^4} = -\frac{3!}{3^4}$

$C_0 = \ln(6)$

$C_1 = \frac{1}{3}$

$C_2 = -\frac{1}{3^2}\cdot\frac{1}{2!}$

$C_3 = +\frac{1}{3^3}\cdot\frac{2!}{3!}$

$C_4 = -\frac{1}{3^4}\cdot\frac{3!}{4!}$

$C_5 = +\frac{1}{3^5}\cdot\frac{4!}{5!}$

for $n \geq 1$:

$C_n = \frac{(-1)^{n-1}}{3^n}\cdot\frac{(n-1)!}{n!}$

$= \frac{(-1)^{n-1}}{3^n}\cdot\frac{1}{n}$

$$\ln(6) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}\cdot\frac{1}{n}(x-3)^n$$

$$= \ln(6) + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3 - \cdots$$

(b) Use the ratio test to find the radius and interval of convergence of the series you found in (c). *No credit will be given for solutions not using the ratio test.*

$$L = \lim_{n\to\infty}\left|\frac{(-1)^n}{3^{n+1}}\cdot\frac{1}{n+1}(x-3)^{n+1}\cdot\frac{3^n\cdot n}{(-1)^{n-1}}\cdot\frac{1}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\underbrace{\frac{n}{n+1}}_{1 \text{ as } n\to\infty}\cdot\frac{(x-3)}{3}\right| = \frac{|x-3|}{3} < 1 \text{ for } |x-3| < 3$$

$-3 < x - 3 < 3$

$0 < x < 6$

ROC $= 3$

IOC $= (0, 6]$

endpts: $x = 0$: $\sum \frac{(-1)^{n-1}(-3)^n}{3^n\cdot n}$

$= \sum\frac{(-1)^{2n-1}}{n} = -\sum\frac{1}{n}$ DIV by p-test

$x = 6$: $\sum\frac{(-1)^{n-1}\cancel{3^n}}{\cancel{3^n}\cdot n} = \sum\frac{(-1)^{n-1}}{n}$ CONV by AST

5. (20 points)

(a) Find the Maclaurin series expansion of the function

$$f(x) = \frac{2e^{\frac{x}{2}} - 2 - x}{x^2},$$

write out the first four nonzero terms, and express the series in sigma notation.

$$2e^{x/2} = 2\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x}{2}\right)^n = 2\sum_{n=0}^{\infty} \frac{1}{n!}\frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{x^n}{2^{n-1}} = 2 + x + \frac{x^2}{4} + \frac{x^3}{3!\cdot 4} + $$

$$2e^{x/2} - 2 - x = \frac{x^2}{4} + \frac{x^3}{3!\cdot 4} + \frac{x^4}{4!\cdot 8} + \frac{x^5}{5!\cdot 16} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n!}\frac{x^n}{2^{n-1}}$$

$$\frac{2e^{x/2} - 2 - x}{x^2} = \boxed{\frac{1}{2\cdot 2!} + \frac{x}{3!\cdot 4} + \frac{x^2}{4!\cdot 8} + \frac{x^3}{5!\cdot 16}} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n!}\frac{x^{n-2}}{2^{n-1}}$$

$$\text{or } = \boxed{\sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+2)!} x^n}$$

(b) What is the value of $f^{(10)}(0)$?

$$f^{(10)}(0) = 10! \cdot c_{10} = 10! \cdot \frac{1}{2^{11}(12)!} = \boxed{\frac{1}{2^{11}\cdot 12\cdot 11}}$$

(c) What is the value of $f^{(11)}(0)$?

$$f^{(11)}(0) = 11!\, c_{11} = 11! \cdot \frac{1}{2^{12}(13)!} = \boxed{\frac{1}{2^{12}\cdot 13\cdot 12}}$$

(d) What is the value of $\lim_{x\to 0} f(x)$?

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1}{2\cdot 2!} + \frac{x}{3!\cdot 4} + \frac{x^2}{4!\cdot 8} + \cdots\right)$$

$$= \boxed{\frac{1}{4}}$$

6. (10 points) Write out the first three terms and then find the sum of each of the following series. *Your table of Maclaurin series expansions might be helpful.*

(a) $\sum_{n=0}^{\infty} \frac{1}{n!\,3^n} 10^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{10}{3}\right)^n = e^{10/3}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2n+1)} \frac{1}{2^{2n+1}} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{2}\right)^{2n+1} = -\arctan\left(\frac{1}{2}\right)$

(c) $\sum_{n=1}^{\infty} \frac{4^n(-1)^{n-1}}{n\,5^n} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{4}{5}\right)^n = \ln\left(1+\frac{4}{5}\right) = \ln\left(\frac{9}{5}\right)$

7. (10 points)

Consider the parametric equations

$$x = \sin(\theta), \qquad y = 1 + \sin^2(\theta)$$

(a) Eliminate the parameter, and write the parametric equations in Cartesian form such that

$y =$ $1 + x^2$

(b) Find bounds for x and y. $-1 \leq \sin\theta \leq 1$ and $0 \leq \sin^2\theta \leq 1$ so

$-1 \leq x \leq 1$

$1 \leq y \leq 2$