Math 143: Calculus III
Midterm I
February 17, 2022

Name: ___________________________ (Please print clearly)

UR ID: ___________________________

Indicate the lecture time you are registered for with a check in the appropriate box:

☐ MW 10:25–11:40am (Cook)

☐ TR 2:00-3:15pm (Sahay)

Instructions:

• You have 180 minutes to work on this exam. You are responsible for checking that this exam has all 10 pages. **Please do not remove any pages.**

• Write your final answers in the provided answer boxes.

• No calculators, phones, electronic devices, books, or notes are allowed during the exam, except for the provided formula sheet.

• Show all work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• Numerical or algebraic simplifications of answers are not required, except when specifically stated otherwise.

• Please write your UR ID in the space provided at the top of each page.

Pledge of Honesty
I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ________________________________
**Unit circle:** The coordinates of the endpoints satisfy \((x, y) = (\cos \theta, \sin \theta)\), where \(\theta\) is the corresponding angle.
1. (15 points) Determine whether the following sequences converge or diverge. If they converge, find their limit. If they diverge, determine whether they diverge to $+\infty$, $-\infty$, or because they oscillate.

(a) \[
\left\{ \frac{(-1)^n}{n^2} \right\}_{n=1}^{\infty}
\]

$$-\frac{1}{n^2} \leq \frac{(-1)^n}{n^2} \leq \frac{1}{n^2}$$

$$\lim_{n \to \infty} \frac{(-1)^n}{n^2} = 0 \quad \text{by the Squeeze Theorem}$$

(b) \[
\left\{ (-n)^2 \right\}_{n=1}^{\infty}
\]

$$\lim_{n \to \infty} (-n)^2 = \lim_{n \to \infty} n^2 = \infty$$

sequence diverges to $\infty$

(c) \[
\left\{ \frac{(-1)^n(n+1)^2}{n} \right\}_{n=1}^{\infty}
\]

for $n$ even, terms approach $\infty$

for $n$ odd, terms approach $-\infty$

the sequence diverges because it oscillates
2. (10 points) Determine whether the following sequences converge or diverge. If they converge, find their limit. If they diverge, determine whether they diverge to $+\infty$, $-\infty$, or because they oscillate.

(a) 
\[
\{\cos(n\pi)\}_{n=1}^{\infty}
\]

for $n$ even: $\cos(n\pi) = 1$
for $n$ odd: $\cos(n\pi) = -1$

sequence diverges because it oscillates

(b) 
\[
\left\{\frac{\cos(n\pi)}{2^n}\right\}_{n=1}^{\infty}
\]

\[-\frac{1}{2^n} \leq \frac{\cos(n\pi)}{2^n} \leq \frac{1}{2^n}\]

by the Squeeze Theorem,

\[
\lim_{n \to \infty} \frac{\cos(n\pi)}{2^n} = 0
\]
3. (10 points) Determine whether the following series converge or diverge. If they converge, find their sum.

(a) 
\[ \sum_{n=1}^{\infty} \frac{5^n}{6^{n-1}} \]

geometric series with \( r = \frac{25}{6} \), divergent

(b) 
\[ \sum_{n=0}^{\infty} \frac{2^{n+1}}{\pi^n} \]

geometric series, \( r = \frac{2}{\pi} \)

\( |r| < 1 \)

series converges to \( \frac{2}{1 - 2/\pi} \)
4. (15 points) Determine whether the following series converge or diverge. Show all work and name any test you use.

(a) 
\[ \sum_{n=1}^{\infty} \frac{n}{n+1} \]

\[ \lim_{n \to \infty} \frac{n}{n+1} = 1 \]

diverges by the divergence test

(b) 
\[ \sum_{n=1}^{\infty} \frac{n^3 - 1}{n^5 + 1} \]

\[ \frac{n^3 - 1}{n^5 + 1} \leq \frac{n^3}{n^5 + 1} \leq \frac{n^3}{n^5} = \frac{1}{n^2} \]

\[ \sum_{n=1}^{8} \frac{1}{n^2} \] is a convergent p-series, so by the direct comparison test, the series converges.
5. (10 points) Use the integral test to determine if the following series converges or diverges. You must use the integral test to get full credit.

*Hint:* \( \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \)

\[
f(x) = \frac{\sqrt{x}}{1+x^{3/2}}
\]

is positive, decreasing, continuous on \([1,\infty)\).

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}
\]

\[
\int_{1}^{\infty} \frac{\sqrt{x}}{1+x^{3/2}} \, dx = \int_{2}^{\infty} \frac{2}{3} \cdot \frac{1}{u} \, du
\]

\[
u = 1+x^{3/2}
\]
\[
du = \frac{3}{2} x^{1/2} \, dx
\]

\[
= \lim_{t \to \infty} \left( \frac{2}{3} \ln(t) - \frac{2}{3} \ln(2) \right)
\]

\[
= \infty
\]

The integral is divergent, so by the integral test, the series is divergent.
6. (10 points) Use the direct comparison test to determine if the following series converge or diverge by comparing it to a geometric series or $p$-series. You must use the direct comparison test to get full credit.

\[
\sum_{n=1}^{\infty} \frac{2^n}{(3n)^n}
\]

\[
\frac{2^n}{(3n)^n} \leq \frac{2^n}{3^n}, \quad \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \text{ is a convergent geometric series}
\]

\[
\text{by DCT, the series converges}
\]
7. (15 points) Consider the following sequence, assuming that the pattern continues:

\[ \{a_n\} = \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \ldots \right\} \]

(a) Find an equation for the \(n\)th term \(a_n\).

\[ a_n = \frac{n}{2^n} \]

(b) Evaluate \(\lim_{n \to \infty} a_n\).

\[ \lim_{n \to \infty} \frac{n}{2^n} = \lim_{n \to \infty} \frac{1}{2^n \ln(2)} = 0 \]

(c) Is the series \(\sum_{n=1}^{\infty} a_n\) a geometric series?

\(\text{NO: look at the ratio } \frac{a_{n+1}}{a_n} = \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \)

\[ \text{the common ratio is not constant} \]
8. (15 points) Suppose a series $\sum_{n=1}^{\infty} a_n$ has partial sums given by

$$S_N = \frac{N}{N+1}.$$ 

(a) What is $\sum_{n=1}^{5} a_n$?

$$\sum_{n=1}^{5} a_n = \frac{5}{6} - \frac{4}{5} = \frac{25}{30} - \frac{24}{30} = \frac{1}{30}$$

(b) What is the value of $a_5$?

$$a_5 = S_5 - S_4 = \frac{5}{6} - \frac{4}{5} = \frac{1}{30}$$

(c) Does $\sum_{n=1}^{\infty} a_n$ converge? If so, what is its sum?

$$\sum_{n=1}^{8} a_n = \lim_{n \to \infty} S_n = 1$$

The series converges to 1.