1. (10 points) Determine whether the following sequences converge or diverge. If they converge, find their limit. If they diverge, state whether they diverge to $+\infty$, $-\infty$ or because they oscillate. **Justify and show all your work.**

(a)

$$a_n = \left(1 + \frac{1}{2n}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{1}{2}}$$

$$= \left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{1}{2}} = e^{1/2}$$

continuous function theorem

$\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e$ and $2n\to\infty$ as $n\to\infty$

ANSWER:

(b)

$$a_n = \frac{2^n}{n3^n}$$

Solution ①: $\lim_{n\to\infty}\frac{2^n}{n3^n} = \lim_{n\to\infty}\frac{(2/3)^n}{n} = \lim_{n\to\infty}\frac{\ln\left(\frac{2}{3}\right)(2/3)^n}{1} = 0$

type $\frac{\infty}{\infty}$ l'HR

since $\left(\frac{2}{3}\right)<1$ so $\left(\frac{2}{3}\right)^n\to 0$ as $n\to\infty$

Solution ②: $0 \le a_n \le \frac{2^n}{3^n}$ and $\lim_{n\to\infty}\frac{2^n}{3^n} = 0$ since $\frac{2}{3}<1$ so $a_n\to 0$ as well by squeeze theorem

ANSWER:

2. (10 points) Determine whether the following sequences converge or diverge. If they converge, find their limit. If they diverge, state whether they diverge to $+\infty$, $-\infty$ or because they oscillate. **Justify and show all your work.**

(a)

$$a_n = \cos\left(\frac{\ln(n)}{n}\right)$$

$$\lim_{n\to\infty} \cos\left(\frac{\ln(n)}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\ln(n)}{n}\right) = \cos(0) = 1$$

since $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n} \underset{\substack{\uparrow \\ \text{type } \frac{\infty}{\infty} \\ \text{l'HR}}}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$

ANSWER:

(b)

$$a_n = \frac{2^n}{n^n} = \frac{2\cdot 2\cdot 2 \cdots 2}{n\cdot n\cdot n \cdots n} = \frac{2}{n}\cdot\Big(\underbrace{\frac{2}{n}\cdots\frac{2}{n}}_{\le 1}\Big)$$

so $0 \le a_n \le \frac{2}{n}$ and $\frac{2}{n} \to 0$ as $n\to\infty$

So, by the squeeze theorem, $a_n \to 0$ as $n\to\infty$ as well

Sorry!

ANSWER:

3. (10 points) Determine whether the following series converges or diverges. If it converges, find its sum. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{7^{2n}}{24^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{24}\left(\frac{49}{24}\right)^n \quad \text{geometric}$$

$$|r| = \left|\frac{49}{24}\right| > 1 \quad \text{so } \boxed{\text{DIV by GST}}$$

ANSWER:

4. (10 points) Determine whether the following series converges or diverges. If it converges, find its sum. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right) \quad \text{telescoping}$$

partial sums: $S_1 = \cos(1) - \cos\left(\frac{1}{2}\right)$

$S_2 = \cos(1) - \cancel{\cos\left(\frac{1}{2}\right)} + \cancel{\cos\left(\frac{1}{2}\right)} - \cos\left(\frac{1}{3}\right)$

$S_3 = \cos(1) - \cancel{\cos\left(\frac{1}{2}\right)} + \cancel{\cos\left(\frac{1}{2}\right)} - \cancel{\cos\left(\frac{1}{3}\right)} + \cancel{\cos\left(\frac{1}{3}\right)} - \cos\left(\frac{1}{4}\right)$

$\vdots$

$S_k = \cos(1) - \cos\left(\frac{1}{k+1}\right)$

sum: $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\cos(1) - \cos\left(\frac{1}{k+1}\right)\right) = \cos(1) - \cos\left(\lim_{k\to\infty}\frac{1}{k+1}\right)$

$= \cos(1) - \cos(0) = \boxed{\cos(1) - 1}$ $\boxed{\text{CONV}}$

ANSWER:

5. (10 points) Determine whether the following series converges or diverges. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{3^n+1}{2^n-n} = \Sigma a_n$$

Solution ①:

① pick $b_n = \frac{3^n}{2^n}$ so Σb_n DIV by GST w/ $r = \frac{3}{2} > 1$

② $\frac{3^n+1}{2^n-n} > \frac{3^n}{2^n} > 0$ (numerator of LHS is larger, denominator of LHS is smaller)

③ So, by CT, Σa_n $\boxed{\text{DIV}}$ as well

Solution ②:

① pick $b_n = \frac{3^n}{2^n}$ so Σb_n DIV by GST w/ $r = \frac{3}{2} > 1$

② $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n+1}{2^n-n} \cdot \frac{2^n}{3^n} = \lim_{n\to\infty} \frac{1+1/3^n}{1-n/2^n} = 1$ nonzero const

(note $\lim_{n\to\infty} n/2^n \underset{\substack{\uparrow \\ \text{type } \frac{\infty}{\infty} \\ \text{LHR}}}{=} \lim_{n\to\infty} \frac{1}{\ln(2)\cdot 2^n} = 0$)

③ So, by LCT, Σa_n $\boxed{\text{DIV}}$ as well

ANSWER:

6. (10 points) Determine whether the following series converge or diverge. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}-6} = \sum a_n$$

① pick $b_n = \frac{1}{n^{1.2}}$ so $\sum b_n$ conv by p-test w/ $p = 1.2 > 1$

② $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\arctan(n)}{n^{1.2}-6} \cdot \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{\arctan(n)}{1 - 6/n^{1.2}}$

$= \frac{\pi}{2}$ nonzero constant ✓

③ So, by LCT, $\sum a_n$ $\boxed{\text{conv}}$ as well

ANSWER:

7. (10 points) Determine whether the following series converge or diverge. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3 + 5n} = \Sigma a_n$$

① pick $b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{2.5}}$ so Σb_n CONV by p-test w/ $p = 2.5 > 1$

② $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^3+5n} \cdot \frac{n^3}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+5n} = \lim_{n\to\infty} \frac{1}{1+5/n^2} = 1$

nonzero const ✓

③ So, by LCT, Σa_n CONV as well.

ANSWER:

8. **(10 points)** Determine whether the following series converge or diverge. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{3n}{n+1} = \sum a_n$$

$$\lim_{n\to\infty} \frac{3n}{n+1} = 3 \neq 0 \text{ so } \sum a_n$$

$\boxed{\text{DIV}}$ by test for div.

ANSWER:

9. (10 points) Use the integral test to determine whether the following series converges or diverges. **To get full credit you must use the integral test.**

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$$

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u}\,du$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \ln|u| \Big]_{\ln(2)}^{\ln(b)} = \lim_{b\to\infty} \left(\ln|\ln(b)| - \ln|\ln(2)| \right)$$

$$= \infty \quad \boxed{\text{DIV}}$$

ANSWER:

10. (10 points) Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$. $= \sum_{n=1}^{\infty} (-1)^n a_n$

(a) Use the Alternating Series Test to show the series converges.

$$\left.\begin{array}{l} \bullet \text{ alt } \checkmark \quad a_n = \frac{1}{n^4} > 0 \\ \bullet \text{ decreasing } \checkmark \quad a_{n+1} = \frac{1}{(n+1)^4} < \frac{1}{n^4} = a_n \\ \bullet \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^4} = 0 \checkmark \end{array}\right\} \Rightarrow \text{conv by AST}$$

ANSWER:

(b) How many terms does it require to approximate the sum with error $\leq .001$?

$\frac{1}{1000} = .001 \leq a_6 = \frac{1}{1296}$ So it requires 5 terms.

n	n^4
1	1
2	16
3	81
4	256
5	625
6	1296

ANSWER:

(c) Approximate the sum of the series to within in .001. (Write it as a single fraction.)

$$-1 + \frac{1}{16} - \frac{1}{81} + \frac{1}{256} - \frac{1}{625} = -\frac{12960000}{12960000} + \frac{810000}{12960000}$$

$$- \frac{160000}{12960000} + \frac{50625}{12960000} - \frac{20736}{12960000} = \frac{-1228011}{12960000}$$

ANSWER: