Math 143 Summer Session A-2 (Midterm II Practice 2 solutions).

1).

a) Alternating series test:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{3/2}}{n^2-6} = \sum_{n=1}^{\infty} (-1)^n b_n, \quad \text{where } b_n = \frac{n^{3/2}}{n^2-6}$$

need to check:

1) $b_n \geq 0$

2) b_n decreasing

3) $\lim_{n\to\infty} b_n = 0$

1) $n^{3/2} \geq 0$ for all $n \geq 0$. $n^2-6 \geq 0$ when $n^2 \geq 6 \Rightarrow n \geq \sqrt{6}$

Take $n \geq 3$. Then, $b_n \geq 0$.

2) Compare to $f(x) = \dfrac{x^{3/2}}{x^2-6}$. using quotient rule,

$$f'(x) = \frac{3/2\, x^{1/2}(x^2-6) - x^{3/2}(2x)}{(x^2-6)^2} = \frac{\frac{3}{2}x^{5/2} - 9x^{1/2} - 2x^{5/2}}{(x^2-6)^2}$$

$$= \frac{-9x^{1/2} - \frac{1}{2}x^{5/2}}{(x^2+6)^2}.$$

The numerator is negative for all $x>0$
The denominator is positive for all x

$\Rightarrow f'(x) < 0$ for all $x > 0 \Rightarrow b_n$ is decreasing for $n > 0$.

3) $$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^{3/2}}{n^2-6} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}}\left(\frac{1}{n^{1/2} - 6/n^{3/2}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^{1/2} - \frac{6}{n^{3/2}}} = 0$$

AST $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^{3/2}}{n^2-6}$ converges! $\longrightarrow$

1a) continued: check absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n^{3/2}}{n^2-6} \right| = \sum_{n=1}^{\infty} \frac{n^{3/2}}{|n^2-6|} \leftarrow \text{converges if and only if } \sum_{n=3}^{\infty} \frac{n^{3/2}}{|n^2-6|} \text{ conv}$$

for $n \geq 3$, $n^2-6 > 0$, so this becomes

$\sum_{n=3}^{\infty} \frac{n^{3/2}}{n^2-6}$. Next, we can do a direct comparison.

$$n^2-6 \leq n^2 \Rightarrow \frac{1}{n^2} \leq \frac{1}{n^2-6} \quad \text{for all } n \geq 3.$$

$$\Rightarrow \frac{n^{3/2}}{n^2} \leq \frac{n^{3/2}}{n^2-6} \Rightarrow \frac{1}{n^{1/2}} \leq \frac{n^{3/2}}{n^2-6} \quad (n \geq 3)$$

By the p-test, $\sum_{n=3}^{\infty} \frac{1}{n^{1/2}}$ diverges $(p = \frac{1}{2} < 1)$

DCT $\Rightarrow \sum_{n=3}^{\infty} \frac{n^{3/2}}{n^2-6}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^n n^{3/2}}{n^2-6} \right|$ diverges

Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n n^{3/2}}{n^2-6}$ is conditionally convergent

1b)

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}.$$

First, we can check for absolute convergence:

$\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$. Notice that $0 \le \left|\frac{\sin(n)}{n^2}\right| = \frac{|\sin(n)|}{n^2} \le \frac{1}{n^2}$.

We may use a direct comparison:

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by a p-test ($p = 2 > 1$)

DCT $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ converges!

$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges absolutely. Moreover,

absolute convergence $\Rightarrow$ convergence, so $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

also converges! So, $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ both converges,

and absolutely converges.

2) $\sum_{n=1}^{\infty} (-1)^n \frac{3\ln(n)}{n}$. AST will be applied again!

$= \sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \frac{3 \cdot \ln(n)}{n}$

1) $3 \cdot \ln(n) \geq 0$ for $n \geq 1$. $n \geq 0$ for $n \geq 0$. So, for $n \geq 1$

$b_n = \frac{3 \cdot \ln(n)}{n} \geq 0$.

2) To show decreasing, take $f(x) = \frac{3\ln(x)}{x}$. Then,

$$f'(x) = \frac{3/x \cdot x - 3\ln(x) \cdot (1)}{x^2} = \frac{3 - 3\ln(x)}{x^2} = \frac{3(1 - \ln(x))}{x^2}$$

$3(1 - \ln(x)) \leq 0$ for $x \geq 1$. x^2 is positive for all $x \neq 0$.

$\Rightarrow f'(x) < 0$ for all $x > 1 \Rightarrow b_n = \frac{3 \cdot \ln(n)}{n}$ is decreasing

for ~~[illegible]~~ $n > 1$.

3) $$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3 \cdot \ln(n)}{n} \overset{\text{L'Hopital}}{=\!=} \lim_{n\to\infty} \frac{3 \cdot \frac{1}{n}}{1} = \lim_{n\to\infty} \frac{3}{n} = 0$$

AST $\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{3\ln(n)}{n}$ converges

However, we still need to check absolute convergence!

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{3\ln(n)}{n} \right| = \sum_{n=1}^{\infty} \frac{3\ln(n)}{n}$, as $\frac{3\ln(n)}{n} \geq 0$ for $n \geq 1$

for $n > e$, $\ln(n) > 1$. We then can use another Direct Comparison →
(or for $n \geq 3$)

2a continued

$$\frac{3\ln(n)}{n} \geq \frac{3}{n} \geq \frac{1}{n} \text{ for } n \geq 3.$$

$\sum_{n=3}^{\infty} \frac{1}{n}$ diverges by a p-test $(p = 1 \leq 1)$.

DCT $\Rightarrow \sum_{n=3}^{\infty} \frac{3\ln(3)}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \left| (-1)^n \frac{3\ln(n)}{n} \right|$ <u>diverges</u>

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{3\ln(n)}{n}$ <u>converges conditionally</u>.

2b) $\sum_{n=1}^{\infty} (-1)^n \frac{3\ln(n)}{n^2}$. We can show this is absolutely convergent via a DCT (again!)

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{3\ln(n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{3\ln(n)}{n^2}, \text{ as } \frac{3\ln(n)}{n^2} \geq 0 \text{ for } n \geq 1.$$

Next, since $\lim_{n\to\infty} \frac{\ln(n)}{n^{1/2}} \overset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{1/n}{\frac{1}{2} n^{-1/2}} = 2 \cdot \lim_{n\to\infty} \frac{n^{1/2}}{n} = 0$

and $\ln(n), n^{1/2} > 0$ for $n > 1$, There is some n_0 such that $\ln(n) \leq n^{1/2}$ for $n \geq n_0$. Therefore for $n \geq n_0$,

$$0 \leq \frac{3\ln(n)}{n^2} \leq \frac{3n^{1/2}}{n^2} = \frac{3}{n^{3/2}}.$$

$\sum_{n=n_0}^{\infty} \frac{3}{n^{3/2}}$ converges by a p-test $(p = 3/2 > 1)$

DCT $\Rightarrow \sum_{n=n_0}^{\infty} \frac{3\ln(n)}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{3\ln(n)}{n^2}$ converges.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n) \cdot 3}{n^2}$ converges absolutely $\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{3\ln(n)}{n^2}$ both converges + converges absolutely!

3) Find radius + Interval of conv:

$$\sum_{n=3}^{\infty} \frac{(-5)^n (x-3)^n}{(n-2)^{3/2} 4^n}$$

Note: Series should start at $n=3$

Proceed via the ratio test:

$$\lim_{n\to\infty} \left| \frac{\left(\frac{(-5)^{n+1} (x-3)^{n+1}}{(n+1-2)^{3/2} 4^{n+1}} \right)}{\left(\frac{(-5)^n (x-3)^n}{(n-2)^{3/2} 4^n} \right)} \right| = \lim_{n\to\infty} \left| \frac{(-5)^{n+1} (x-3)^{n+1}}{(n-1)^{3/2} 4^{n+1}} \cdot \frac{4^n (n-2)^{3/2}}{(-5)^n (x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| -\frac{5}{4} (x-3) \cdot \left(\frac{n-2}{n-1} \right)^{3/2} \right| = \frac{5}{4} |x-3| \cdot \underbrace{\lim_{n\to\infty} \left(\frac{n-2}{n-1} \right)^{3/2}}_{=1}$$

so the limit $= \frac{5}{4}|x-3|$. For absolute convergence, we need $\frac{5}{4}|x-3| < 1 \Rightarrow |x-3| < 4/5 \Rightarrow$ Radius of convergence $= 4/5$

R =

⚠ Need to check edge cases: $|x-3| = 4/5 \longrightarrow$

i) $x - 3 = 4/5 \Rightarrow x = \frac{4}{5} + 3 \Rightarrow x = \frac{19}{5}$

$$\hookrightarrow \sum_{n=3}^{\infty} \frac{(-5)^n}{4^n} \frac{(4/5)^n}{(n-2)^{3/2}} = \sum_{n=3}^{\infty} \frac{(-1)^n}{(n-2)^{3/2}}.$$

AST; $b_n = \frac{1}{(n-2)^{3/2}}$

positive $(n \geq 2)$

decreasing: If $f(x) = \frac{1}{(x-2)^{3/2}}$

$f'(x) = \frac{-3}{2}(x-2)^{-5/2} < 0$ for $x \geq 2$

$\lim_{n\to\infty} \frac{1}{(n-2)^{3/2}} = 0$

AST $\Rightarrow$ convergence

ii) $x - 3 = -4/5$

$$\hookrightarrow \sum_{n=3}^{\infty} \frac{(-5)^n}{4^n} \frac{\left(-\frac{4}{5}\right)^n}{(n-2)^{3/2}} = \sum_{n=3}^{\infty} \frac{1}{(n-2)^{3/2}}$$

LCT with $\frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\frac{1}{(n-2)^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}}{(n-2)^{3/2}} = 1 \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-test $\Rightarrow \sum_{n=3}^{\infty} \frac{1}{(n-2)^{3/2}}$ conv by LCT $\Rightarrow$

ROC $= 4/5$,

IOC $= [3 - 4/5, 3 + 4/5]$

$= [11/5, 19/5]$

4) $f(x) = \ln(2x)$. Power series expansion around $x=3$

a)

Will look like $\sum_{n=0}^{\infty} C_n (x-3)^n$.

First, let $y = x-3$, so $x = y+3$. Then,

$$\ln(2x) = \ln(2(y+3)) \leftarrow \text{Let } g(y) = \ln(2(y+3))$$

• We can find a power series representation of $g(y)$ centered at 0:

$$\frac{d}{dy} g(y) = \frac{1}{2(y+3)} \cdot 2 = \frac{1}{y+3} = \frac{1}{3(1+y/3)} = \frac{1}{3} \frac{1}{1-(-y/3)}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-y/3)^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n y^n}{3^n}, \text{ provided that } |-y/3| = |y/3| < 1$$

$$\Rightarrow |y| < 3.$$

Next:

$$g(y) = \int \frac{d}{dy} g(y)\, dy = \int \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n y^n}{3^n} dy = \frac{1}{3} \sum_{n=0}^{\infty} \int \frac{(-1)^n y^n}{3^n} dy = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n y^{n+1}}{(n+1)3^n} + C$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n y^{n+1}}{(n+1)3^{n+1}}$$

$$g(0) = \ln(2(0+3)) = \ln(6) = C.$$

$$\Rightarrow g(y) = \ln(6) + \sum_{n=0}^{\infty} \frac{(-1)^n y^{n+1}}{(n+1)3^{n+1}} = \ln(6) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^n}{n \cdot 3^n}.$$

Letting $y = x-3$, we get

$$f(x) = g(x-3) = \ln(6) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-3)^n}{n \cdot 3^n}.$$

4b) For convergence, we can get rid of the constant term and look at $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-3)^n}{n \cdot 3^n}$.

Ratio test: $\lim_{n\to\infty} \left| \frac{\frac{(-1)^n (x-3)^{n+1}}{(n+1)3^{n+1}}}{\frac{(-1)^{n-1}(x-3)^n}{n \cdot 3^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n (x-3)^{n+1}}{(n+1)\, 3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^{n-1}(x-3)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(x-3)}{3} \cdot \frac{n}{n+1} \right| = \frac{|x-3|}{3} \cdot \overbrace{\lim_{n\to\infty} \frac{n}{n+1}}^{=1} = \frac{|x-3|}{3}$

for conv w/ratio test: $\frac{|x-3|}{3} < 1 \Rightarrow |x-3| < 3.$

$\Rightarrow R =$ radius of convergence $= 3$.

CHECK EDGE CASES:

i) $|x-3| = 3; \quad x-3 = 3 \quad \Rightarrow x = 6$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by AST: $b_n = \frac{1}{n}$

i) $\frac{1}{n} \geq 0$ for $n \geq 0$

ii) $n+1 > n$ (decreas

$\Rightarrow \frac{1}{n} > \frac{1}{n+1}$

iii) $\lim_{n\to\infty} \frac{1}{n} = 0$.

ii) $x - 3 = -3 \Rightarrow x = 0$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n}$

$= \sum_{n=1}^{\infty} \frac{-1}{n} \rightarrow$ diverges by ~~p-~~ test $(p = 1 \leq 1.)$

$\Rightarrow$ IOC $= (0, 6]$.

ROC $= 3$

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