1. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use and justify its use.*

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}\cos(n)}{n^3-2} = \sum a_n$$

① First, check for absolute convergence:

$$|a_n| = \left|\frac{\sqrt{n}\cos(n)}{n^3-2}\right| \leq \frac{\sqrt{n}}{n^3-2} \text{ for } n \geq 2 \quad (\text{since } |\cos(n)| \leq 1)$$

So $\sum |a_n|$ converges if $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3-2}$ converges by CT.

② pick $b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{2.5}}$. $\sum b_n$ converges by p-test with $p = 2.5 > 1$

③ LCT: $\lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{n^3-2} \cdot \frac{\cancel{n^{2.5}}}{1} = \lim_{n\to\infty} \frac{n^3}{n^3-2} = 1$ non-zero const.

So $\sum \frac{\sqrt{n}}{n^3-2}$ converges as well by LCT.

④ Since, by ① and ③, $\sum |a_n|$ converges, $\sum a_n$ is $\boxed{\text{abs. conv}}$

[Thus, there is no need to check for conditional convergence.]

2. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use and justify its use.*

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n} = \sum a_n$$

① First, check for absolute convergence:

$$|a_n| = \left|\frac{(-1)^n \ln(n)}{n}\right| = \frac{\ln(n)}{n} > \frac{1}{n} \quad \text{for } n \geq 3$$

So, since $\sum \frac{1}{n}$ div. by p-test w/ $p = 1 \leq 1$,

$\sum \frac{\ln(n)}{n}$ div. as well by p-test.

Thus, $\sum |a_n|$ DIV so $\sum a_n$ is NOT abs. conv.

② Next, check for conditional convergence using AST:

- $\sum a_n$ is alternating ✓
- $f(x) = \frac{\ln(x)}{x} \Rightarrow f'(x) = \frac{\frac{1}{x} \cdot x - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2} < 0$ for $x > e$.

  so a_n is decreasing ✓
- $\lim\limits_{n\to\infty} \frac{\ln(n)}{n} \underset{\substack{\text{type } \frac{\infty}{\infty} \\ \text{L'HR}}}{=} \lim\limits_{n\to\infty} \frac{1/n}{1} = 0$ ✓

So $\sum a_n$ converges by AST

③ By ① and ②, $\sum a_n$ is $\boxed{\text{conditionally convergent}}$

3. (15 points) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-4)^n(x+1)^n}{9^n\sqrt{n}}.$$

ratio test: $\lim_{n\to\infty} \left| \frac{(-4)^{n+1}(x+1)^{n+1}}{9^{n+1}\sqrt{n+1}} \cdot \frac{9^n\sqrt{n}}{(-4)^n(x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{4}{9}(x+1)\sqrt{\frac{n}{n+1}} \right|$

$= \frac{4}{9}|x+1| < 1$

when $|x+1| < \boxed{\frac{9}{4} = R}$

so the series converges when $|x+1| < \frac{9}{4}$ and diverges for $|x+1| > \frac{9}{4}$, and the ratio test is inconclusive when $|x+1| = \frac{9}{4}$.

Solve $|x+1| < \frac{9}{4}$: $-\frac{9}{4} < x+1 < \frac{9}{4}$

$-\frac{13}{4} < x < \frac{5}{4}$

check endpoints: $x = -\frac{13}{4}$: $\sum_{n=1}^{\infty} \frac{(-4)^n\left(-\frac{9}{4}\right)^n}{9^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIV by p-test with $p = \frac{1}{2} \leq 1$

$x = \frac{5}{4}$: $\sum_{n=1}^{\infty} \frac{(-4)^n\left(\frac{9}{4}\right)^n}{9^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ CONV by AST

- alt ✓
- decr. ✓
- lim = 0 ✓

So the radius of convergence $R = \frac{9}{4}$

and interval of convergence $IOC = \left(-\frac{13}{4}, \frac{5}{4}\right]$

4. (15 points) Consider the function $f(x) = e^{-x}$.

(a) Find the Taylor series of $f(x)$ about $x = 3$. Write out the first three nonzero terms, and express the series in sigma notation.

$f(x) = e^{-x}$ $\quad f(3) = e^{-3}$ $\quad c_0 = e^{-3}$

$f'(x) = -e^{-x}$ $\quad f'(3) = -e^{-3}$ $\quad c_1 = -e^{-3}$

$f''(x) = e^{-x}$ $\quad f''(3) = e^{-3}$ $\quad c_2 = e^{-3}/2!$

$f'''(x) = -e^{-x}$ $\quad f'''(3) = -e^{-3}$ $\quad c_3 = -e^{-3}/3!$

$f^{(4)}(x) = e^{-x}$ $\quad f^{(4)}(3) = e^{-3}$ $\quad c_4 = e^{-3}/4!$

$\vdots$

$c_n = (-1)^n e^{-3}/n!$

$$\sum_{n=0}^{\infty} \frac{(-1)^n e^{-3}}{n!}(x-3)^n = e^{-3} - e^{-3}(x-3) + \frac{e^{-3}}{2}(x-3)^2 - \cdots$$

(b) Use the ratio test to find the radius and interval of convergence of the series you found in (a). *No credit will be given for solutions not using the ratio test.*

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} e^{-3}}{(n+1)!}(x-3)^{n+1} \cdot \frac{(n)!}{(-1)^{n} e^{-3}(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{-(x-3)}{n+1} \right| = 0 < 1$$

for all x

so ROC $= \infty$

IOC $= (-\infty, \infty)$

5. (15 points)

(a) Find the Maclaurin series expansion of the function

$$f(x) = \frac{\cos(x^2) - 1}{x^4}.$$

Write out the first four nonzero terms, and express the series in sigma notation.

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^2)^{2n} = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \frac{(x^2)^8}{8!} - \cdots$$

$$\cos(x^2) - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} = -\frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots$$

$$\frac{\cos(x^2) - 1}{x^4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n-4} = -\frac{1}{2} + \frac{x^4}{4!} - \frac{x^8}{6!} + \frac{x^{12}}{8!} - \cdots$$

(b) What is the value of $f^{(12)}(0)$?

$$f^{(n)}(a) = c_n \cdot n! \quad \text{so} \quad f^{(12)}(0) = \boxed{\frac{1}{8!} \cdot 12!}$$

(c) What is the value of $f^{(11)}(0)$?

$c_{11} = 0$ since x^{11} doesn't occur in the above series expansion; Hence, $f^{(11)}(0) = c_{11} \cdot 11! = \boxed{0}$

(d) What is the value of $\lim_{x \to 0} f(x)$?

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(-\frac{1}{2} + \frac{x^4}{4!} - \frac{x^8}{6!} + \frac{x^{12}}{8!} - \cdots \right) = \boxed{-\frac{1}{2}}$$

6. (15 points) Write out the first two terms and then find the sum of each of the following convergent series. *You do not need to show the series are convergent. Your table of Maclaurin series expansions might be helpful.*

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(-2)^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(-\frac{1}{2}\right)^n \underset{\substack{\uparrow \\ x=\frac{-1}{2}}}{=} \ln\left(1-\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right)$ ✓

first two terms: $-\frac{1}{2} - \frac{1}{8} + \cdots$

(b) $\sum_{n=0}^{\infty} \frac{(-5)^n}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-5/2)^n}{n!} \underset{\substack{\uparrow \\ x=-\frac{5}{2}}}{=} e^{-5/2}$

first two terms: $1 + \frac{-5}{2} + \cdots$

(c) $\sum_{n=0}^{\infty} \frac{5(-1)^{n-1}3^{2n+1}}{(2n+1)4^{2n+1}} = -5 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{3}{4}\right)^{2n+1} \underset{\substack{\uparrow \\ x=\frac{3}{4}}}{=} -5\arctan\left(\frac{3}{4}\right)$

first two terms: $-5\left(\frac{3}{4}\right) + 5\left[\frac{(3/4)^3}{3}\right]$

7. (20 points) Consider the parametric equations for a curve $C(\theta)$ defined for all θ by

$$x = 5\cos(2\theta), \qquad y = 2\sin(2\theta)$$

(a) Eliminate the parameter, and write the resulting Cartesian equation in the form given below. *No credit will be given for solutions not showing any work.*

$$\frac{y^2}{4} =$$

$$\cos^2(2\theta) + \sin^2(2\theta) = 1$$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\boxed{\frac{y^2}{4} = 1 - \frac{x^2}{25}}$$

(b) Find parametric equations $x = f(t)$ and $y = g(t)$ for a circle of radius 2 centered at the origin together with an interval of t-values such that the circle is traced out once in the counterclockwise direction starting at $(x, y) = (2, 0)$ at $t = 0$.

$$\left\{ \begin{array}{l} x = 2\cos(t) \\ y = 2\sin(t) \end{array} \right\} \qquad 0 \le t \le 2\pi$$

Common Taylor series centered at $x = 0$:

Function	Taylor Series	Initial Terms	Converges for
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + x^4 + \cdots$	$-1 < x < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$1 - x + x^2 - x^3 + x^4 - \cdots$	$-1 < x < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	All x
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\tan^{-1}(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$-1 \le x \le 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 < x \le 1$