**1. (10 points)** Determine whether the following sequences converge or diverge. If they converge, find their limit. If they diverge, state whether they diverge to $+\infty$, $-\infty$ or because they oscillate. **Justify and show all your work.**

(a)

$$a_n = (-1)^n \sin\left(\frac{13}{n}\right)$$

$$|a_n| = \left|(-1)^n \sin\left(\frac{13}{n}\right)\right| = \left|\sin\left(\frac{13}{n}\right)\right|$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\sin\left(\frac{13}{n}\right)\right| \underset{\uparrow}{=} \left|\sin\left(\lim_{n\to\infty}\frac{13}{n}\right)\right| = |\sin(0)| = 0$$

CFT

By the corollary to the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$ as well.

$\boxed{\text{CONV to 0}}$

(b)

$$a_n = (-4)^n$$

$$|a_n| = 4^n \text{ so } \lim_{n\to\infty} |a_n| = \lim_{x\to\infty} 4^x = \infty$$

but $a_n$ is positive if $n$ is even and negative if $n$ is odd so

$\{a_n\}$ $\boxed{\text{DIV b/c OSCL.}}$

**2. (10 points)** Determine whether the following sequences converge or diverge. If they converge, find their limit. If they diverge, state whether they diverge to $+\infty$, $-\infty$ or because they oscillate. **Justify and show all your work.**

(a)

$$a_n = \frac{n^2}{e^{2n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{e^{2n}} \underset{\substack{\text{type } \frac{\infty}{\infty} \\ \text{LHR}}}{=} \lim_{n\to\infty} \frac{2n}{2e^{2n}} \underset{\substack{\text{type } \frac{\infty}{\infty} \\ \text{LHR}}}{=} \lim_{n\to\infty} \frac{2}{4e^{2n}} = 0$$

$\boxed{\text{CONV to } 0}$

(b)

$$a_n = \ln\left(\frac{7n-7}{6n+4}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{7n-7}{6n+4}\right) \overset{\text{CFT}}{=} \ln\left[\lim_{n\to\infty}\left(\frac{7n-7}{6n+4}\right)\right]$$

$$= \ln\left[\lim_{n\to\infty}\left(\frac{7-7/n}{6+4/n}\right)\right] = \ln\left(\frac{7}{6}\right)$$

CONV to $\frac{7}{6}$

**3. (10 points)** Determine whether the following series converges or diverges. If it converges, find its sum. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=0}^{\infty} \frac{3^n + 4^n}{3^{2n}} = \sum a_n$$

$$\sum a_n = \sum_{n=0}^{\infty} \frac{3^n}{3^{2n}} + \sum_{n=0}^{\infty} \frac{4^n}{3^{2n}} = \sum_{n=0}^{\infty} \left(\frac{3}{9}\right)^n + \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n$$

both are conv. by GST

$a = 1$, $r = \frac{1}{3}$   $a = 1$, $r = \frac{4}{9}$

$$= \frac{1}{1 - \frac{1}{3}} + \frac{1}{1 - 4/9}$$

$$= \frac{3}{2} + \frac{9}{5} = \frac{15 + 18}{10} = \frac{33}{10}$$

**4. (10 points)** Consider the telescoping series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left[ \ln\left(\frac{1}{n}\right) - \ln\left(\frac{1}{n+1}\right) \right]$$

(a) Find the first three partial sums.

$s_1 = \ln(1) - \ln(\frac{1}{2})$

$s_2 = \ln(1) - \ln(\frac{1}{2}) + \ln(\frac{1}{2}) - \ln(\frac{1}{3})$

$s_3 = \ln(1) - \ln(\frac{1}{2}) + \ln(\frac{1}{2}) - \ln(\frac{1}{3}) + \ln(\frac{1}{3}) - \ln(\frac{1}{4})$

(b) Find a formula for the $k^{th}$ partial sum $s_k = \sum_{n=1}^{k} \ln\left(\frac{1}{n}\right) - \ln\left(\frac{1}{n+1}\right)$.

$S_k = \ln(1) - \ln(\frac{1}{k+1}) = -\ln(\frac{1}{k+1}) = \ln(k+1)$

(c) Determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, find its sum.

$\boxed{\text{DIV}}$ because $\sum a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} -\ln(\frac{1}{k+1}) = \infty$

**5. (10 points)** Use the integral test to determine whether the following series converges or diverges. **To get full credit you must use the integral test.**

$f(x) = 3x^2 e^{-x^3}$ is cont's ✓
decr ✓
pos. ✓

$$\sum_{n=1}^{\infty} 3n^2 e^{-n^3}$$

$$\int_1^{\infty} 3x^2 e^{-x^3} dx = \lim_{b\to\infty} \int_1^b 3x^2 e^{-x^3} dx \underset{\substack{u = x^3 \\ du = 3x^2 dx}}{=} \lim_{b\to\infty} \int_1^{b^3} e^{-u} du$$

$$= \lim_{b\to\infty} -e^{-u}\Big]_1^{b^3} = \lim_{b\to\infty} \left( -e^{-b^3} + e^{-1} \right) = e^{-1}$$

$\boxed{\text{CONV}}$

So, by the integral test, $\sum_{n=1}^{\infty} 3n^2 e^{-n^3}$ CONV as well.

**6. (10 points)** Determine whether the following series converges or diverges. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2 - \sqrt{n}} = \sum a_n$$

① pick $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{1.5}}$ so $\sum b_n$ CONV by p-test w/ $p = 1.5 > 1$

② $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2 - \sqrt{n}} \cdot \frac{n^2}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^2}{n^2 - \sqrt{n}} = \lim_{n\to\infty} \frac{1}{1 - 1/n^{1.5}} = 1$ nonzero const.

③ so, by the LCT, $\sum a_n$ $\boxed{\text{CONV}}$ as well

**7. (10 points)** Determine whether the following series converge or diverge. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} 3^{1/n}$$

$$\lim_{n\to\infty} 3^{1/n} \underset{\substack{\uparrow \\ \text{CFT}}}{=} 3^{\lim_{n\to\infty} 1/n} = 3^0 = 1 \neq 0$$

so $\sum 3^{1/n}$ $\boxed{\text{DIV}}$ by

Test for Divergence

**8. (10 points)** Determine whether the following series converge or diverge. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

AST: alt ✓ $\frac{1}{n!} > 0$

decr. ✓ $\frac{1}{(n+1)!} = \frac{1}{n+1} \cdot \frac{1}{n!} < \frac{1}{n!}$

lim ✓ $\lim_{n\to\infty} \frac{1}{n!} = 0$ since $n! > n$ so $n! \to \infty$.

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ CONV by AST

**9. (10 points)** Determine whether the following series converges or diverges. **Justify and show all your work. Name any test you are using.**

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n+5}$$

Solution ① First, $\sum \frac{\ln(n)}{n}$ DIV by CT since $\frac{\ln(n)}{n} > \frac{1}{n}$ for $n \geq 3$ and $\sum \frac{1}{n}$ DIV by p test w/ $p = 1 \leq 1$.

Second, $\lim_{n\to\infty} \frac{\ln(n)}{n+5} \cdot \frac{n}{\ln(n)} = \lim_{n\to\infty} \frac{n}{n+5} = \lim_{n\to\infty} \frac{1}{1+5/n} = 1$ nonzero const

so $\sum \frac{\ln(n)}{n+5}$ DIV as well, by LCT.

Solution ② First, $\sum \frac{1}{n+5}$ DIV by LCT since $\lim_{n\to\infty} \frac{1}{n+5} \cdot \frac{n}{1} = 1$ nonzero const and $\sum \frac{1}{n}$ DIV (harmonic)

Second, $\sum \frac{\ln(n)}{n+5}$ DIV by CT since $\frac{\ln(n)}{n+5} > \frac{1}{n+5}$ for $n \geq 3$.

**10. (10 points)** Consider the convergent alternating series $\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n$.

(a) How many terms does it require to approximate the sum with error $\leq \frac{1}{64} = 0.015625$?

$a_{k+1} \leq \frac{1}{64}$ for $k+1 = 4$ so $k = 3$

$\Rightarrow$ it requires 3 terms

(b) Approximate the sum of the series to within in $\frac{1}{64}$. (Write it as a single fraction.)

$1 - \frac{1}{4} + \frac{1}{16} = \frac{16}{16} - \frac{4}{16} + \frac{1}{16} = \frac{13}{16}$

(c) What is the sum of infinite series?

$a = 1$

$r = -\frac{1}{4}$

$\frac{1}{1-(-1/4)} = \frac{1}{5/4} = \frac{4}{5}$