Solutions week 9

Wednesday, October 28, 2020 10:15 PM

Warm up:

① ...

② (a) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = 2$ (b) $\sum_{n=0}^{\infty} \frac{(1-2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n = \frac{1}{1-(-1/3)} = \frac{1}{4/3} = \frac{3}{4}$

If x is not in the interval of convergence, then $\sum_{n=0}^{\infty} x^n$ will diverge.

③ $[0,\infty)$ is not an interval of convergence, as 0 and ∞ do not have the same distance to any $a \in \mathbb{R}$.

Other intervals of convergence can be of the form: $[a-R, a+R)$
$[a-R, a+R]$
$(a-R, a+R)$

Problems:

① (a) $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n \cdot 3^n}\right|} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \cdot \sqrt[n]{\left(\frac{|x-2|}{3}\right)^n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \cdot \frac{|x-2|}{3} = \frac{|x-2|}{3}$ $\boxed{\text{Root Test}}$

- Divergence for $\frac{|x-2|}{3} > 1$
- Absolute Convergence for $\frac{|x-2|}{3} < 1$, or equivalently, for $-1 < x < 5$.
- $\sum_{n=0}^{\infty} \frac{((-1)-2)^n}{n \cdot 3^n} = \sum_{n=0}^{\infty} \frac{1}{n} \cdot \left(\frac{-3}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges for $x=1$
- $\sum_{n=0}^{\infty} \frac{((5)-2)^n}{n \cdot 3^n} = \sum_{n=0}^{\infty} \frac{1}{n} \cdot \left(\frac{3}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges for $x=5$.

So the radius of convergence is $R = \frac{5-(-1)}{2} = \frac{6}{2} = 3$, and the interval of convergence is $[-1, 5)$.

(b) $\lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty} \frac{x^{n+1}\, n!}{x^n\,(n+1)!} = \lim_{n\to\infty} \frac{\cancel{x^n} \cdot x \cdot \cancel{n!}}{\cancel{x^n} \cdot (n+1) \cdot \cancel{n!}} = \lim_{n\to\infty} \frac{x}{n+1} = 0$ $\boxed{\text{Ratio Test}}$

Absolute Convergence for all $x \in \mathbb{R}$, so the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

(c) $\lim_{n\to\infty} \left|\frac{\frac{(n+1)!\,(x+4)^{n+1}}{\sqrt{n+1}}}{\frac{n!\,(x+4)^n}{\dots}}\right| = \lim_{n\to\infty} \frac{(n+1)!\,|(x+4)^{n+1}| \cdot \sqrt{n}}{n!\,|(x+4)^n| \cdot \sqrt{n+1}} = \lim_{n\to\infty} \frac{(n+1)\cdot \cancel{n!}\,|x+4|\,\cancel{|(x+4)^n|}\,\sqrt{n}}{\cancel{n!}\,\cancel{|(x+4)^n|} \cdot \sqrt{n+1}} = \lim_{n\to\infty} (n+1)\,|x+4| \cdot \sqrt{\frac{n}{n+1}}$

(c) $\lim_{n\to\infty}\left|\dfrac{\frac{(n+1)!\,(x+4)^{n+1}}{\sqrt{n+1}}}{\frac{n!\,(x+4)^n}{\sqrt{n}}}\right| = \lim_{n\to\infty} \dfrac{(n+1)!\,|(x+4)^{n+1}|\cdot\sqrt{n}}{n!\,|(x+4)^n|\cdot\sqrt{n+1}} = \lim_{n\to\infty}\dfrac{(n+1)\cdot\cancel{n!}\,|x+4|\,\cancel{|(x+4)^n|}\,\sqrt{n}}{\cancel{n!}\,\cancel{|(x+4)^n|}\cdot\sqrt{n+1}} = \lim_{n\to\infty}(n+1)\,|x+4|\cdot\sqrt{\dfrac{n}{n+1}}$

We know $\lim_{n\to\infty}\sqrt{\dfrac{n}{n+1}} = 1$ and $\lim_{n\to\infty} n+1 = \infty$, so $\lim_{n\to\infty}(n+1)|x+4|\sqrt{\dfrac{n}{n+1}} = \infty$

unless $|x+4| = 0$.

Thus, the radius of convergence is $R=0$ and the interval of convergence is $\{-4\}$.

(d) $\lim_{n\to\infty}\sqrt[n]{\left|\dfrac{n(x+1)^n}{4^n}\right|} = \lim_{n\to\infty}\sqrt[n]{n}\cdot\sqrt[n]{\left(\dfrac{|x+1|}{4}\right)^n} = \lim_{n\to\infty}\sqrt[n]{n}\cdot\dfrac{|x+1|}{4} = \dfrac{|x+1|}{4}.$

- Divergence for $\dfrac{|x+1|}{4} > 1$
- Absolute Convergence for $\dfrac{|x+1|}{4} < 1$, or equivalently, for $-5 < x < 3$
- $\sum_{n=0}^{\infty}\dfrac{n((-5)+1)^n}{4^n} = \sum_{n=0}^{\infty} n\cdot\left(\dfrac{-4}{4}\right)^n = \sum_{n=1}^{\infty}(-1)^n\cdot n$ diverges for $x=-5$
- $\sum_{n=0}^{\infty}\dfrac{n((3)+1)^n}{4^n} = \sum_{n=0}^{\infty} n\cdot\left(\dfrac{4}{4}\right)^n = \sum_{n=0}^{\infty} n$ diverges for $x=3$

Therefore, the radius of convergence is $R = \dfrac{3-(-5)}{2} = \dfrac{8}{2} = 4$ and the interval of convergence is $(-5,3)$.

(2)
- $\sum_{n=1}^{\infty}\dfrac{(x-5)^n}{n}$ has interval of convergence $[4,6)$ (you need to check that's true)
- $\sum_{n=1}^{\infty}\dfrac{(-1)^n(x-5)^n}{n}$ has interval of convergence $(4,6]$ (you need to check that's true)
- $\sum_{n=1}^{\infty}\dfrac{(x-5)^n}{n^2}$ has interval of convergence $[4,6]$ (you need to check that's true)

(3) (a) $\lim_{n\to\infty}\left|\dfrac{\frac{x^{n+1}}{1\cdot3\cdot5\cdot7\cdot\cdots\cdot(2(n+1)-1)}}{\frac{x^n}{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}}\right| = \lim_{n\to\infty}\dfrac{|x|}{2(n+1)-1} = 0 \quad\Rightarrow R=\infty$

(b) $\lim_{n\to\infty}\left|\dfrac{\frac{(n+1)!\,x^{n+1}}{1\cdot3\cdot5\cdot7\cdot\cdots\cdot(2(n+1)-1)}}{\frac{n!\,x^n}{1\cdot3\cdot5\cdot7\cdot\cdots\cdot(2n-1)}}\right| = \lim_{n\to\infty}\dfrac{(n+1)\cdot|x|}{2(n+1)-1} = |x|\cdot\lim_{n\to\infty}\dfrac{n+1}{2n+1} = \dfrac{|x|}{2} \quad\Rightarrow R=2.$

(b) $\lim_{n\to\infty}\left|\frac{\frac{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2(n+1)-1)}{n!\,x^n}}{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}\right| = \lim_{n\to\infty}\frac{(n+1)\cdot|x|}{2(n+1)-1} = |x|\cdot\lim_{n\to\infty}\frac{n+1}{2n+1} = \frac{|x|}{2} \Rightarrow R=2.$

④ No! For example, for $c_n = \left(\frac{-1}{4}\right)^n\cdot\frac{1}{n}$, $\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\frac{4^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ converges, but $\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\cdot\frac{(-4)^n}{n} = \sum_{n=0}^{\infty}\frac{1}{n}$ doesn't.

The second one, yes! If $\sum_{n=0}^{\infty} c_n 4^n$ converges, then (by the Ratio Test). $\lim_{n\to\infty}\left|\frac{c_{n+1}4^{n+1}}{c_n\cdot 4^n}\right| = 4\cdot\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \le 1.$

That is, $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \le \frac{1}{4}$. In particular, $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < \frac{1}{2}$, so $\lim_{n\to\infty}\left|\frac{c_{n+1}(-2)^{n+1}}{c_n(-2)^n}\right| < 1$

and by the Ratio Test $\sum_{n=0}^{\infty} c_n(-2)^n$ converges.