Solutions week 8

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Warm up:

① $\lim_{n\to\infty} \sqrt[n]{n} = ?$ Let $y = \sqrt[n]{n}$, then $\ln(y) = \ln(n^{1/n}) = \frac{1}{n}\ln(n)$.

Now, $\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{\ln(n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore, by continuity of e^x, $\lim_{n\to\infty} y = e^{\lim_{n\to\infty}\ln(y)} = e^0 = 1$.

$\therefore \lim_{n\to\infty} \sqrt[n]{n} = 1$.

② $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)3^{n+1}}{4^{n+2}}}{\frac{n3^n}{4^{n+1}}}\right| = \lim_{n\to\infty} \frac{(n+1)}{n}\cdot\frac{3}{4} = \frac{3}{4}\lim_{n\to\infty}\frac{n+1}{n} = \frac{3}{4}\cdot 1 = \frac{3}{4}$

Since $\frac{3}{4} < 1$, then the Ratio Test says $\sum_{n=1}^{\infty} \frac{n3^n}{4^{n+1}}$ converges absolutely.

③ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n3^n}{4^{n+1}}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{n3^n}{4^{n+1}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\,\sqrt[n]{3^n}}{\sqrt[n]{4}\,\sqrt[n]{4^n}} = \lim_{n\to\infty} \frac{3}{4}\sqrt[n]{\frac{n}{4}}$.

Let $y = \sqrt[n]{n/4} \Rightarrow \ln(y) = \frac{1}{n}\ln(n/4)$

Since $\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{\ln(n/4)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n/4}\cdot\frac{1}{4}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$,

then $\lim_{n\to\infty} \sqrt[n]{n/4} = \lim_{n\to\infty} e^{\ln(y)} = e^0 = 1$.

Therefore,

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{3}{4}\lim_{n\to\infty}\sqrt[n]{n/4} = \frac{3}{4}\cdot 1 = \frac{3}{4}$.

By the Root Test, since $\frac{3}{4} < 1$, then $\sum_{n=1}^{\infty} \frac{n3^n}{4^{n+1}}$ is absolutely convergent.

Problems:

① (a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)^p}{1/n^p}\right| = \lim_{n\to\infty} \frac{n^p}{(n+1)^p}\cdot\frac{1/n^p}{1/n^p} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^p} = \lim_{n\to\infty} \frac{1}{(1+1/n)^p} = 1$.

It doesn't depend on p but it is inconclusive.

(b) No, if $a_n = \frac{p(n)}{q(n)}$, where $p(n)$ and $q(n)$ are polynomials in n, then

$p(n+1)/$

(b) No, if $u_n = \frac{p(n)}{q(n)}$, where $p(n)$ and $q(n)$ are polynomials in n, then

$$\frac{a_{n+1}}{a_n} = \frac{p(n+1)/q(n+1)}{p(n)/q(n)} = \frac{p(n+1)q(n)}{p(n)q(n+1)}$$

Note that the degree of the numerator equals the degree of the denominator, and the leading coefficient in both cases is the product of the leading coefficients of p and q. Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{\text{leading coeff. of } p(n+1)q(n)}{\text{Leading coeff. of } p(n)q(n+1)} = 1.$$

(c) $$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{|1/n^p|} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{n^p}} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{n}^{\,p}} \overset{\text{continuity}}{=} \frac{1}{(\lim_{n\to\infty}\sqrt[n]{n})^p} = \frac{1}{1^p} = 1.$$

(d) $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

$p(k) = a_n k^n + a_{n-1}k^{n-1} + \cdots + a_2 k^2 + a_1 k + a_0$

$$y = \sqrt[k]{|p(k)|} \Longrightarrow \ln y = \ln\left(|p(k)|^{1/k}\right) = \frac{1}{k}\ln(|p(k)|)$$

$$\Longrightarrow \lim_{k\to\infty}\ln y = \lim_{k\to\infty}\frac{\ln(|p(k)|)}{k} \overset{L'H}{=} \lim_{n\to\infty}\frac{\pm\frac{1}{p(k)}\cdot p'(k)}{1}$$

$$= \lim_{k\to\infty}\frac{na_n k^{n-1} + (n-1)a_{n-1}k^{n-2} + \cdots + 2a_2 k + a_1}{a_n k^n + a_{n-1}k^{n-1} + \cdots + a_2 k^2 + a_1 k + a_0}$$

$= 0$, as the denominator has higher degree.

$$\Rightarrow \lim_{k\to\infty}\sqrt[k]{|p(k)|} = e^0 = 1$$

$\therefore$ The test is inconclusive.

Trying the ratio test would be a waste of time because this series is of the form $\frac{p(n)}{q(n)}$, where $q(n)\equiv 1$. But we know $p(n)$ is unbounded, so $\sum p(n)$ diverges.

(e) (i) • Ratio Test inconclusive by part (a)

• Root Test inconclusive by part (c)

(ii) • $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2/2^{n+1}}{2/2^n}\right| = \lim_{n\to\infty}\frac{2^n}{2^{n+1}} = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$ conclusive! (abs. conv.)

• $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{2/2^n} = \lim \frac{\sqrt[n]{2}}{2} = \lim \frac{\sqrt[n]{2}}{2} = \frac{0}{} = 0.$ Conclusive! (abs.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2/2^{n+1}}{2/2^n}\right| = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|2/2^n|} = \lim_{n\to\infty} \frac{\sqrt[n]{2}}{\sqrt[n]{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2}}{2} = \frac{0}{2} = 0$. conclusive! (abs. conv.)

(iii). Square roots are powers 1/2, so all the analysis we did before still holds for this case: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$ quotient of leading coeff. $= 1$. inconclusive!

- $\lim_{n\to\infty} \sqrt[n]{\left|\frac{\sqrt{n}}{1+n^2}\right|} = \lim_{n\to\infty} \frac{\sqrt[2n]{n}}{\sqrt[n]{1+n^2}} = \frac{e^0}{e^0} = 1$

$y = \sqrt[2n]{n} \Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(n)}{2n} = \lim_{n\to\infty} \frac{1}{2n} = 0$

$y = \sqrt[n]{1+n^2} \Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(1+n^2)}{n} = \lim_{n\to\infty} \frac{\frac{2n}{1+n^2}}{n}$

$= \lim_{n\to\infty} \frac{2}{1+n^2} = 0$

inconclusive!

② $a_1 = 2$, $a_{n+1} = \frac{5n+1}{4n+3} a_n = \left(\frac{5n+1}{4n+3}\right)\left(\frac{5n+1}{4n+3}\right) a_{n-1} = \cdots = \left(\frac{5n+1}{4n+3}\right)^n \cdot a_1 = 2\left(\frac{5n+1}{4n+3}\right)^n$.

$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2\left(\frac{5n+1}{4n+3}\right)^n = 2\sum_{n=1}^{\infty} \left(\frac{5n+1}{4n+3}\right)^n$.

$\lim_{n\to\infty} \sqrt[n]{\left|\frac{5n+1}{4n+3}\right|^n} = \lim_{n\to\infty} \left|\frac{5n+1}{4n+3}\right| = \lim_{n\to\infty} \frac{5n+1}{4n+3} = \frac{5}{4} > 1$.

The Root Test says that the series diverges.

③ (a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}}\right| = \lim_{n\to\infty} \frac{(n+1)!\ 100^n}{n!\ 100^{n+1}} = \lim_{n\to\infty} \frac{(n+1)}{100} = +\infty$

By the Ratio Test, $\sum \frac{n!}{100^n}$ diverges.

(b) $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1-n}{2+3n}\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1-n}{2+3n}\right|^n} = \lim_{n\to\infty} \frac{|1-n|}{|2+3n|} = \lim_{n\to\infty} \frac{n-1}{2+3n} = \frac{1}{3} < 1$

By the Root Test, $\sum \left(\frac{1-n}{2+3n}\right)^n$ is absolutely convergent.

(c) $\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{10n} \cdot \left(\frac{-9}{10}\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{10n}}\ \sqrt[n]{\left|\frac{9}{10}\right|^n} = \lim_{n\to\infty} \frac{9}{10\sqrt[n]{10n}}$

Now, if $y = \sqrt[n]{10n} \Rightarrow \ln y = \ln((10n)^{1/n}) = \frac{1}{n}\ln(10n)$

$\Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(10n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{10n}\cdot 10}{1} = 0$

$\Rightarrow \lim \sqrt[n]{10n} = e^0 = 1$

$$\lim_{n\to\infty} \qquad \lim_{n\to\infty} \frac{\quad}{n} \qquad \lim_{n\to\infty} \frac{\quad}{1}$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{10n} = e^{0} = 1$$

Therefore, $\displaystyle \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-9)^n}{n\,10^{n+1}}\right|} = \frac{9}{10\cdot 1} = \frac{9}{10} < 1.$

By the Root Test, the series converges absolutely.

(d) $$\lim_{n\to\infty} \left| \frac{(2(n+1))!/((n+1)!)^{n+1}}{(2n)!/(n!)^n} \right| = \lim_{n\to\infty} \frac{(2(n+1))!\,(n!)^n}{(2n)!\,((n+1)!)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)!\,(n!)^n}{(2n)!\,((n+1)!)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}\,(n!)^n}{\cancel{(2n)!}\,((n+1)\cdot n!)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(n!)^n}}{(n+1)^{n+1}\cdot\cancel{(n!)^n}\,(n!)}$$

$$= 0$$

By the Ratio Test, $\sum \frac{(2n)!}{(n!)^n}$ is absolutely convergent.

(e) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}\, \cancel{2\cdot 4\cdot 6}\cdots \cancel{(2n)}(2(n+1))}{\cancel{5\cdot 8\cdot 11}\cdots \cancel{(3n+2)}(3(n+1)+2)}}{\dfrac{\cancel{(-1)^n}\, \cancel{2\cdot 4\cdot 6}\cdots \cancel{(2n)}}{\cancel{5\cdot 8\cdot 11}\cdots \cancel{(3n+2)}}} \right| = \lim_{n\to\infty} \frac{2(n+1)}{3(n+1)+2}$$

$$= \lim_{n\to\infty} \frac{2n+2 \cdot \frac{1}{n}}{3n+5 \cdot \frac{1}{n}} = \lim_{n\to\infty} \frac{2+2/n}{3+5/n} = \frac{2}{3} < 1$$

By the Ratio Test, $\sum \frac{(-1)\,2\cdot 4\cdot 6\cdots(2n)}{5\cdot 8\cdot 11\cdots(3n+2)}$ is absolutely convergent.

arctan(x) > 0 whenever x > 0.

(f) $$\lim_{n\to\infty} \sqrt[n]{|(\arctan(n))^n|} = \lim_{n\to\infty} \sqrt[n]{|\arctan(n)|^n} = \lim_{n\to\infty} \arctan(n) = \pi/2 > 1$$

By the Root Test, $\sum \arctan(n)^n$ diverges.