Solutions week 5

Monday, October 5, 2020 3:50 PM

Warm up:

① $$\int_e^\infty \frac{1}{x(\ln x)^2}\,dx \overset{\substack{u=\ln x\\ du=\frac{1}{x}dx}}{=} \int_1^\infty \frac{du}{u^2} = \lim_{r\to\infty} \left.\frac{-1}{u}\right|_1^r = \lim_{r\to\infty}\left(\cancel{\frac{-1}{r}}^{\,0} + \frac{1}{1}\right) = 1.$$

②. The function $f(x) = \dfrac{2x-1}{3x+1}$ has derivative $f'(x) = \dfrac{2(3x+1)-3(2x-1)}{(3x+1)^2}$

$$= \frac{\cancel{6x}+2-\cancel{6x}+3}{(3x+1)^2} = \frac{5}{(3x+1)^2} > 0.$$

So $f(x)$ is increasing at all x. In particular, it's increasing at all n.

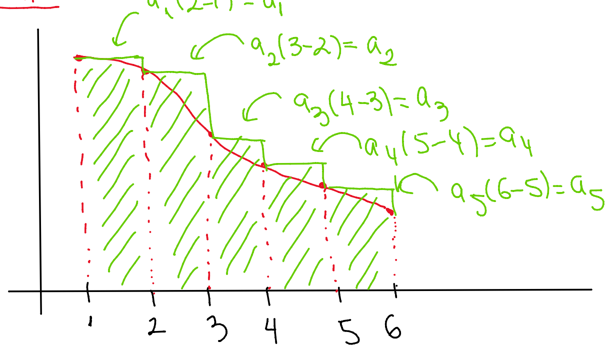
- Continuous whenever $3x+1 \neq 0$; i.e., whenever $x \neq -1/3$. Since all inputs "n" are positive, this is not a problem.

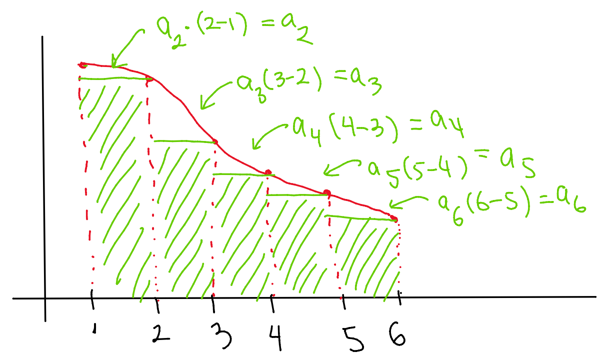
- $x \geq 1 \implies 2x-1 \geq 1$ and $3x+1 \geq 4$, $\forall x$
  So f is positive.

Since not decreasing, it does not satisfy the requirements.

Problems.

①





$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

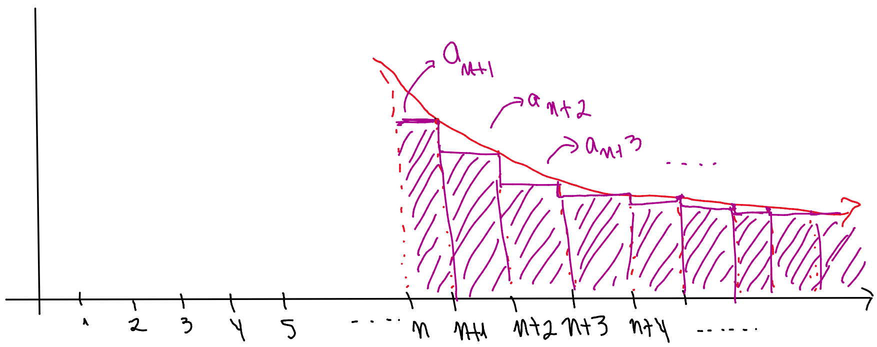
$$\int_1^6 f(x)\,dx < \sum_{n=1}^{5} a_n \qquad\qquad \sum_{n=2}^{6} a_n < \int_1^6 f(x)\,dx$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

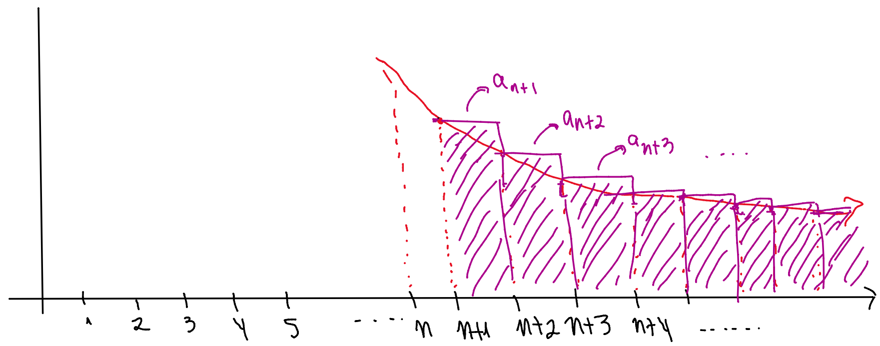
$$\sum^{6} a_n < \int^{6} f(x)\,dx < \sum^{5} a_n.$$

$$\sum_{n=2}^{6} a_n < \int_1^6 f(x)\,dx < \sum_{n=1}^{5} a_n .$$

② (a)



$$\Rightarrow R_n \le \int_n^\infty f(x)\,dx$$



$$\Rightarrow \int_{n+1}^\infty f(x)\,dx \le R_n .$$

$$\therefore \boxed{\int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx}$$

$$\Leftrightarrow \int_{n+1}^\infty f(x)\,dx < S - S_n < \int_n^\infty f(x)\,dx \qquad \Big|\ +S_n$$

$$\Leftrightarrow S_n + \int_{n+1}^\infty f(x)\,dx < S < S_n + \int_n^\infty f(x)\,dx. \qquad //$$

(b) Want $R_n \le \int_n^\infty \frac{1}{x^2}\,dx \le 0.25$ and $\int_n^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_n^\infty = \lim_{r\to\infty} -\frac{1}{r} + \frac{1}{n} = \frac{1}{n}$

So $n \in \mathbb{N}$ must be such that $\frac{1}{n} \le \frac{1}{4}$. Take $n = 4$

Using this, $R_4 \le 0.25$ and $S \approx S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{205}{144}$

with an error of $R_4 \le 0.25$.

③ (a) $\frac{\sqrt{x}+4}{x^2}$ is: • continuous on $(1,\infty)$ as quotient of continuous functions, $x^2 \neq 0$.

• positive, as $\sqrt{x}+4>0$ and $x^2>0$ on $(1,\infty)$.

• decreasing, as $\frac{d}{dx}\left(\frac{\sqrt{x}+4}{x^2}\right) = \frac{\frac{1}{2\sqrt{x}}\cdot x^2 - 2x(\sqrt{x}+4)}{x^4} = \frac{-\frac{3}{2}x^{3/2} - 8x}{x^4} < 0$.

and $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$ converges iff $\sum_{n=2}^{\infty} \frac{\sqrt{n}+4}{n^2}$ converges. By the integral test,

$$\longrightarrow \sum_{n=2}^{\infty} \frac{\sqrt{n}+4}{n^2} \le \int_1^{\infty} \frac{\sqrt{x}+4}{x^2}\,dx = \int_1^{\infty} x^{1/2-2} + 4x^{-2}\,dx = \frac{x^{-1/2}}{-1/2}\Bigg|_1^{\infty} + \frac{4x^{-1}}{-1}\Bigg|_1^{\infty}$$

$$= \lim_{r\to\infty}\left(\cancelto{0}{\frac{-2}{\sqrt{r}}} - \left(\frac{-2}{\sqrt{1}}\right)\right) + \left(\cancelto{0}{\frac{-4}{r}} - \left(\frac{-4}{1}\right)\right) = 6$$

So $\sum_{n=2}^{\infty} \frac{\sqrt{n}+4}{n^2}$ converges and so does $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$.

(b) $\frac{x}{x^4+1}$ is: • continuous, as quotient of continuous functions, $x^4+1 \neq 0$

• positive, as both x and x^4+1 are.

• decreasing, as $\frac{d}{dx}\left(\frac{x}{x^4+1}\right) = \frac{1\cdot(x^4+1) - x(4x^3)}{(x^4+1)^2} = \frac{-3x^4+1}{(x^4+1)^2} \le \frac{-2}{(x^4+1)^2} < 0$.

$$\overset{\text{Int. test}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{n}{n^4+1} \le \int_0^{\infty} \frac{x}{x^4+1}\,dx = \frac{1}{2}\int_{u(0)}^{u(\infty)} \frac{du}{u^2+1} = \frac{1}{2}\arctan(u)\Big|_0^{\infty} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}.$$

$u = x^2$
$du = 2x\,dx$

So $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.