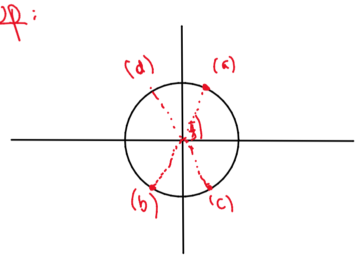
Solutions Week 2

Thursday, September 17, 2020 11:05 AM

Warm up:

1



(a) $(1, \pi/3) = (1, 7\pi/3)$

(b) $(-1, \pi/3) = (1, 4\pi/3)$

(c) $(1, -\pi/3) = (-1, 2\pi/3)$

(d) $(-1, -\pi/3) = (1, 2\pi/3)$

→ I am using this one to draw them in the xy-plane.

2 (a) (i) $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \arctan\left(\frac{x}{y}\right) = \arctan(1) = \frac{\pi}{4}$

$+\pi$ if II or III quadrant

$+2\pi$ if IV quadrant.

$\boxed{(\sqrt{2}, \pi/4)}$

(ii) $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \arctan\left(\frac{x}{y}\right) \underbrace{+\pi}_{\text{since in the second quadrant}} = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$\boxed{(\sqrt{2}, 3\pi/4)}$

(iii) $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \arctan\left(\frac{x}{y}\right) \underbrace{+2\pi}_{\text{since in the 4th quadrant}} = \arctan(-1) + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$\boxed{(\sqrt{2}, 7\pi/4)}$

(iv) $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \arctan\left(\frac{x}{y}\right) \underbrace{+\pi}_{\text{since in the 3rd quadrant}} = \arctan(1) + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

(b) All these points have distance $\sqrt{2}$ to the origin, so they live in the circle

$$(x-0)^2 + (y-0)^2 = (\sqrt{2})^2,$$

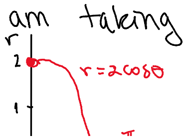
or simply $x^2 + y^2 = 2$.

On polar coordinates, this is given by $r = \sqrt{2}$.

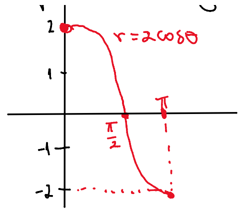
Problems:

1 (I am taking here $r = 2\cos\theta$, $0 \leq \theta \leq \pi$)

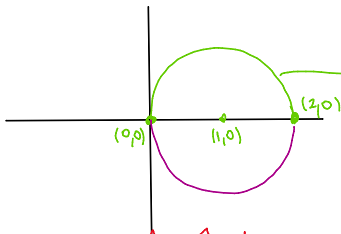
(a)



(a)



(b)



When θ moves from 0 to $\pi/2$, r moves from 2 to 0.

(We are not sure yet if this should look more flat...
But we will know soon.)

When θ moves from $\pi/2$ to π, r moves from 0 to -2

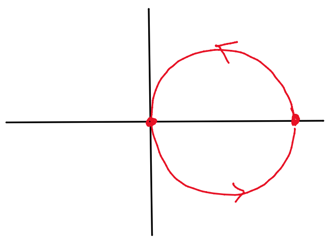
We change r to positive and add π to θ.

$\therefore$ When θ moves from $3\pi/2$ to 2π, r moves from 0 to 2.

(thus, now we are in the 4th quadrant instead of 2nd)

Q: is it actually a circle?

(c)



(d) $x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r} \Rightarrow r = 2\cos\theta = 2\frac{x}{r}$

$$r = \frac{2x}{r} \quad \Big/ \cdot r$$

$\Leftrightarrow \quad r^2 = 2x$, But $r^2 = x^2 + y^2$

$$\Leftrightarrow \quad x^2 + y^2 = 2x$$

$$\Leftrightarrow \quad (x^2 - 2x) + y^2 = 0$$

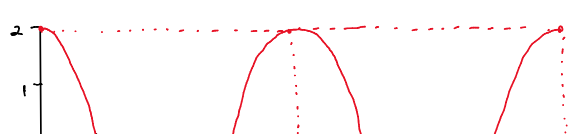
$$\Leftrightarrow \quad (x^2 - 2x + 1) - 1 + y^2 = 0$$

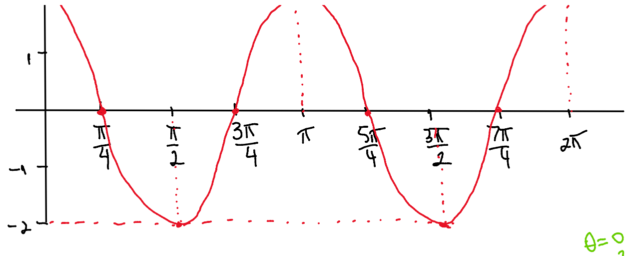
$$\Leftrightarrow \quad (x-1)^2 + (y-0)^2 = 1^2$$

$\longleftarrow$ Equation of the circle centered at $(-1,0)$ and of radius 1

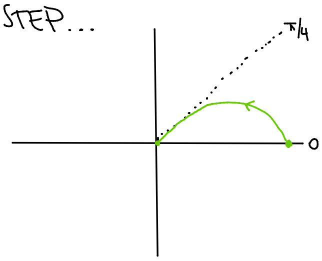
So it actually was a circle!!

② We first graph $r = 2\cos(2\theta)$ for $0 \le \theta \le 2\pi$ on the $r\theta$-plane:



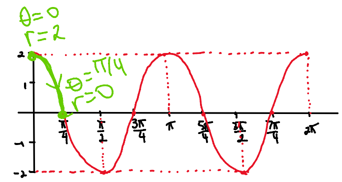


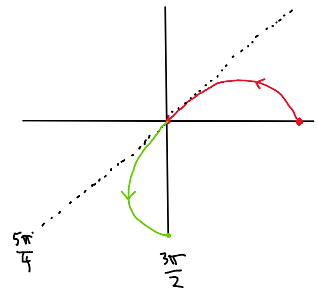
STEP BY STEP...



As $\theta: 0 \to \pi/4$,

$r: 2 \to 0$





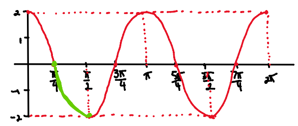
As $\theta: \pi/4 \to \pi/2$,

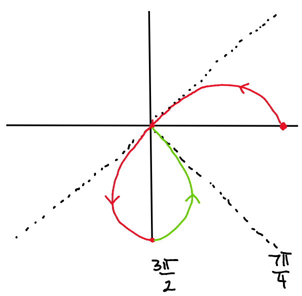
$r: 0 \to -2$

So we change it to

$\theta: 5\pi/4 \to 3\pi/2$

$r: 0 \to 2$





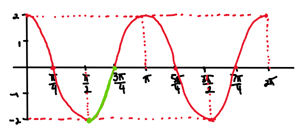
As $\theta: \pi/2 \to 3\pi/4$,

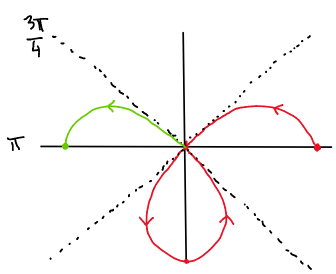
$r: -2 \to 0$

So we change it to

$\theta: 3\pi/2 \to 7\pi/4$,

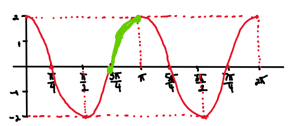
$r: 2 \to 0$

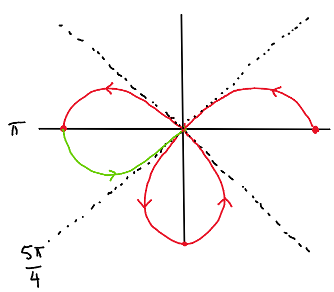




As $\theta: 3\pi/4 \to \pi$,

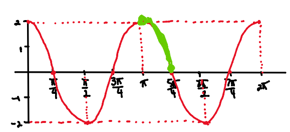
$r: 0 \to 2$



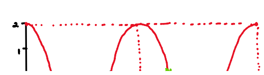


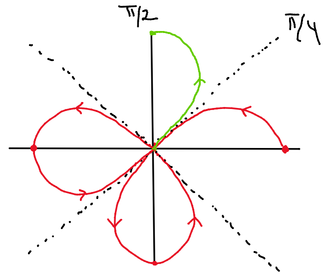
As $\theta: \pi \to 5\pi/4$,

$r: 2 \to 0$



As $\theta: 5\pi/4 \to 3\pi/2$,





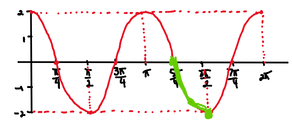
As $\theta : 5\pi/4 \longrightarrow 3\pi/2$,

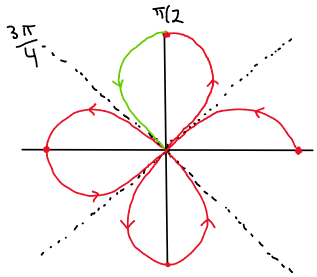
$r : 0 \longrightarrow -2$

so we change it to

$\theta : \pi/4 \longrightarrow \pi/2$

$r : 0 \longrightarrow 2$





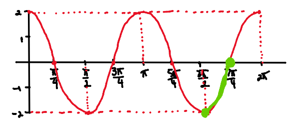
As $\theta : 3\pi/2 \longrightarrow 7\pi/4$,

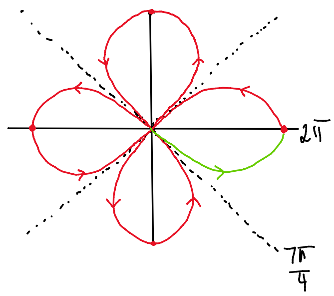
$r : -2 \longrightarrow 0$

So we change it to

$\theta : \pi/2 \longrightarrow 3\pi/4$

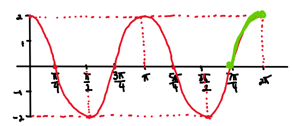
$r : 2 \longrightarrow 0$



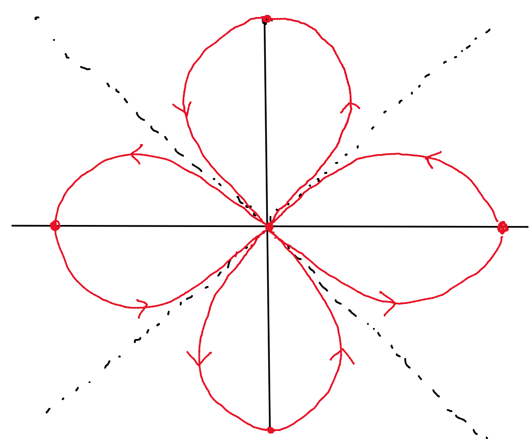


As $\theta : 7\pi/4 \longrightarrow 2\pi$

$r : 0 \longrightarrow 2$



Therefore the final graph of the curve looks like

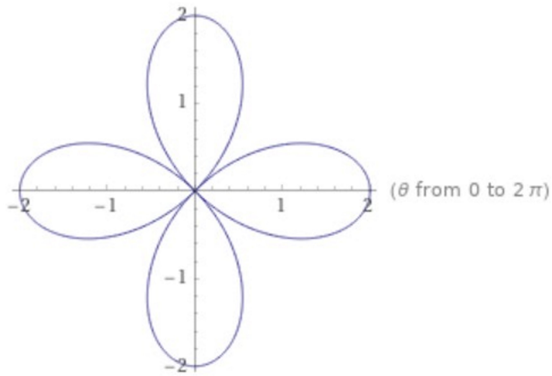


(b)

Input:

$r = 2\cos(2\theta)$

Polar plot:



Not bad, huh?

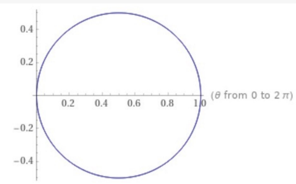
(c)

* Images taken from Wolframalpha.

Input:

$r = \cos(\theta)$

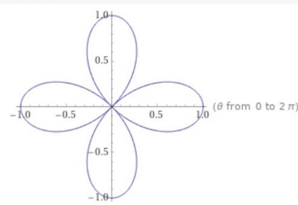
Polar plot:



Input:

$r = \cos(2\theta)$

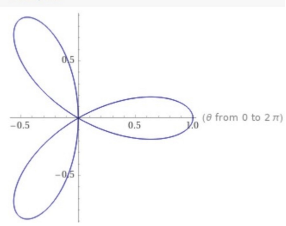
Polar plot:



Input:

$r = \cos(3\theta)$

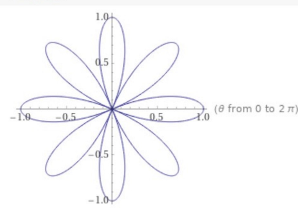
Polar plot:



Input:

$r = \cos(4\theta)$

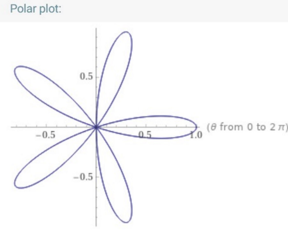
Polar plot:



Input:

$r = \cos(5\theta)$

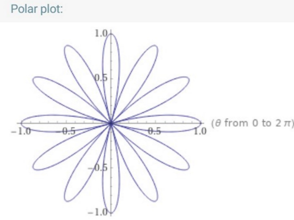
Polar plot:

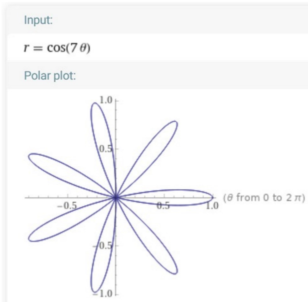


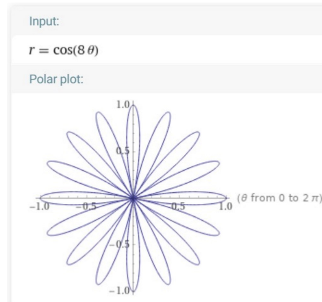
Input:

$r = \cos(6\theta)$

Polar plot:





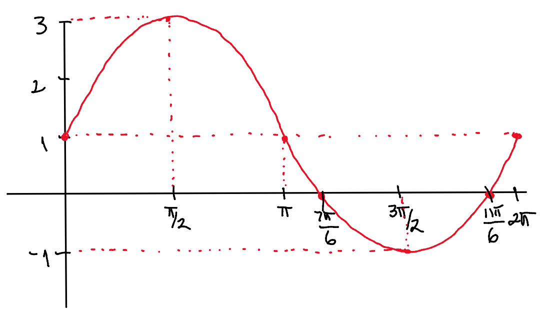


It seems like $r = \cos(n\theta)$, n odd has k petals and it's symmetric with respect to the x-axis.

It seems like $r = \cos(n\theta)$, n even has $2k$ petals and it's symmetric with respect to both the x-axis and the y-axis.

Can you do the same for $r = \sin(n\theta)$? What do you conclude?

3



$1 + 2\sin(0) = 1$

$1 + 2\sin(\pi/2) = 3$

$1 + 2\sin(\pi) = 1$

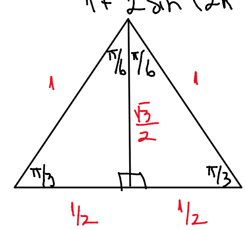
$1 + 2\sin(3\pi/2) = -1$

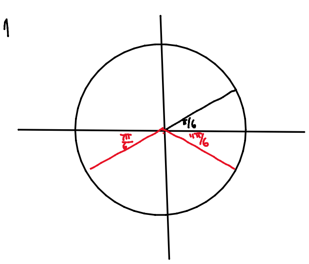
$1 + 2\sin(2\pi) = 1$

Also $1 + 2\sin\theta = 0 \iff \sin\theta = -1/2$

$\iff \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

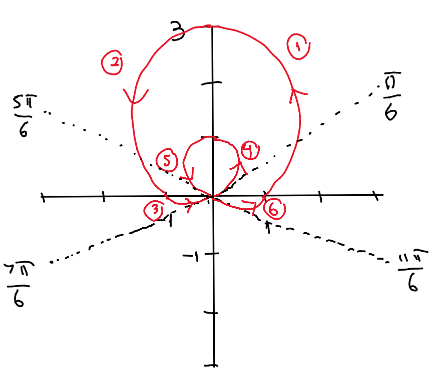
(as $\theta \in [0, 2\pi]$).





$\frac{1}{2} = \sin(\pi/6)$

$\sin\left(\frac{7\pi}{6}\right) = \sin\left(\frac{11\pi}{6}\right) = \frac{-1}{2}$



① $\theta: 0 \to \pi/2$
$r: 1 \to 3$

② $\theta: \pi/2 \to \pi$
$r: 3 \to 1$

③ $\theta: \pi \to 7\pi/6$
$r: 1 \to 0$

④ $\theta: 7\pi/6 \to 3\pi/2$
$r: 0 \to -1$

CHANGE IT!

$\theta: \pi/6 \to \pi/2$

CHANGE IT!

$\theta: \pi/6 \longrightarrow \pi/2$

$r: 0 \longrightarrow 1$

5 $\theta: 3\pi/2 \longrightarrow 11\pi/6$

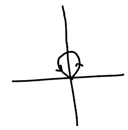
$r: -1 \longrightarrow 0$

CHANGE IT!

$\theta: \pi/2 \longrightarrow 5\pi/6$

6 $\theta: 11\pi/6 \longrightarrow 2\pi$

$r: 0 \longrightarrow 1$

Now let's find the area inside the inner loop 

$$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} r^2\, d\theta = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1+2\sin\theta)^2\, d\theta = \int_{7\pi/6}^{11\pi/6} 1 + 4\sin\theta + 4\sin^2\theta \quad d\theta$$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = 1 - \cos(2\theta)$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} 1 + 4\sin\theta + 2(1-\cos(2\theta))\, d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} 3 + 4\sin\theta - 2\cos(2\theta)\, d\theta$$

$$= \frac{1}{2}\left[3\theta - 4\cos\theta - \sin(2\theta)\right]\Bigg|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

$$= \frac{3(11-7)\pi}{2\cdot 6} - \frac{4}{2}\left(\cos\left(\frac{11\pi}{6}\right) - \cos\left(\frac{7\pi}{6}\right)\right) - \left(\sin\left(\frac{11\pi}{3}\right) - \sin\left(\frac{7\pi}{3}\right)\right)\cdot \frac{1}{2}$$

$$= \pi - 2\left(\frac{\sqrt{3}}{2}\right) + 2\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \pi + \frac{\sqrt{3}}{2}\left(-2 - 2 + 1/2 + 1/2\right)$$

$$= \boxed{\pi - \frac{3\sqrt{3}}{2}}$$