Solutions week 11

Sunday, November 8, 2020 9:29 PM

Warm up:

① (a) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} \frac{n!}{(n-1)!} c_n (x-a)^{n-1}$

$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} = \sum_{n=2}^{\infty} \frac{n!}{(n-2)!} c_n (x-a)^{n-2}$

$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) c_n (x-a)^{n-3} = \sum_{n=3}^{\infty} \frac{n!\, c_n}{(n-3)!} (x-a)^{n-3}$

$f^{(4)}(x) = \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) c_n (x-a)^{n-4} = \sum_{n=4}^{\infty} \frac{n!\, c_n}{(n-4)!} (x-a)^{n-4}$

$f^{(5)}(x) = \sum_{n=5}^{\infty} n(n-1)(n-2)(n-3)(n-4) c_n (x-a)^{n-5} = \sum_{n=5}^{\infty} \frac{n!}{(n-5)!} c_n (x-a)^{n-5}$

(b) $f'(a) = 1 \cdot c_1 \cdot 1 = c_1$

$f''(a) = 2 \cdot (2-1) c_2 \cdot 1 = 2 \cdot c_2$

$f'''(a) = 3 \cdot (3-1)(3-2) c_3 \cdot 1 = 3 \cdot 2 \cdot 1 \cdot c_3 = 6 \cdot c_3$

$f^{(4)}(a) = 4 \cdot (4-1) \cdot (4-2) \cdot (4-3) \cdot c_4 \cdot 1 = 4!\, c_4$

$f^{(5)}(a) = 5!\, c_5$.

(c) $c_0 = \frac{f(a)}{0!}$, $c_1 = \frac{f'(a)}{1!}$, $c_2 = \frac{f''(a)}{2!}$, $c_3 = \frac{f'''(a)}{3!}$, $c_4 = \frac{f^{(4)}(a)}{4!}$

(d) $c_n = \frac{f^{(n)}(a)}{n!}$

② $a_0 = \frac{f(2)}{0!} = 6$

$a_1 = \frac{f'(2)}{1!} = 1$

$a_2 = \frac{f''(2)}{2!} = \frac{3}{2}$

$a_3 = \frac{f'''(2)}{3!} = \frac{-2}{6} = -\frac{1}{3}$

$a_4 = \frac{f^{(4)}(2)}{4!} = \frac{0}{4!} = 0$.

Problem:

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$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

(a) $g(x) = \sum_{n=0}^{\infty} 3x^2 \frac{(-1)^n (x/4)^{2n+1}}{(2n+1)!}$ $R=\infty$ since $3x^2$ and $\sin(x/4)$ have domain $= \mathbb{R}$.

(b) $h(x) = 6x \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{6x^{2n+1}}{n!}$, $R=\infty$ since both $6x$ and e^{x^2} have domain $= \mathbb{R}$.

② (a) $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1} \cdot (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/4)^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

(b) $$\frac{8}{6} + \frac{16}{24} + \frac{32}{120} + \frac{64}{720} + \cdots = \sum_{n=0}^{\infty} \frac{2^{n+3}}{(n+3)!} = \sum_{n=3}^{\infty} \frac{2^n}{n!} = e^2 - (1+2+2) = e^2 - 5$$

(c) $$\sum_{n=1}^{\infty} \frac{1}{n 2^n} = \sum_{n=1}^{\infty} \frac{(1/2)^n}{n} = -\ln(1 - 1/2) = -\ln(1/2) = \ln(2).$$

③ (a) $$\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x\to 0} \frac{x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}{x^3} = \lim_{x\to 0} -\frac{1}{x^3} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$= \lim_{x\to 0} -\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{2n+1} = -\frac{(-1)\cdot 1}{2\cdot 1+1} = \frac{+1}{3}.$$

(b) $$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}{1 + x - \sum_{n=0}^{\infty} \frac{x^n}{n!}} = \lim_{x\to 0} \frac{-\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}{-\sum_{n=2}^{\infty} \frac{x^n}{n!}}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\sum_{n=1}^{\infty} \frac{2n(-1)^n x^{2n-1}}{(2n)!}}{\sum_{n=2}^{\infty} \frac{n x^{n-1}}{n!}} \overset{L'H}{=} \lim_{x\to 0} \frac{\sum_{n=1}^{\infty} \frac{\cancel{2n}\cancel{(2n-1)}(-1)^n x^{2n-2}}{\cancel{(2n)}\cancel{(2n-1)}(2n-2)!}}{\sum_{n=2}^{\infty} \frac{\cancel{n}\cancel{(n-1)} x^{n-2}}{\cancel{n}\cancel{(n-1)}(n-2)!}}$$

$$= \frac{\frac{(-1)\cdot 1}{0!}}{\frac{1}{0!}} = -1 .$$

4 (a) $c_0 = f(2) = \ln(2)$

$$c_1 = \frac{f'(2)}{1!} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$c_2 = \frac{f''(2)}{2!} = \frac{-1\cdot 1}{(2)^2\cdot 2} = -\frac{1}{8}$$

$$c_3 = \frac{f'''(2)}{3!} = \frac{1\cdot 2}{(2)^3\cdot 3\cdot 2} = \frac{1}{24}$$

$$\vdots$$

$$c_n = \frac{(-1)^{n+1}(n-1)!}{2^n\cdot n!}$$

$$\ln(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!\,(x-2)^n}{2^n\cdot n!} = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{2^n\cdot n}$$

(b) We did before (Workshop 10)

$$\ln(2+t) = \ln(2) - \sum_{n=0}^{\infty} \frac{1}{n+1}\cdot\left(-\frac{t}{2}\right)^{n+1}$$

$$\Rightarrow \ln(x) = \ln(2+(x-2)) = \ln(2) - \sum_{n=0}^{\infty} \frac{1}{n+1}\left(-\frac{(x-2)}{2}\right)^{n+1} = \ln(2) + \sum_{n=0}^{\infty} \frac{(-1)^{n+2}(x-2)^{n+1}}{2^{n+1}\cdot(n+1)}$$

$$= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{2^n\cdot n} .$$