Solutions week 10

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Warmup: ① $f(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ← Radius of convergence $R=1$.

② (a) $\dfrac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^{n-1} 3^{n+1} 5x^n\right) = \sum_{n=1}^{\infty} \dfrac{d}{dx}(-1)^{n-1} 3^{n+1} 5x^n$ [additivity of $\frac{d}{dx}$]

* Note now we start from $n=1$ *

$= \sum_{n=1}^{\infty} (-1)^{n-1} 3^{n+1} 5 \dfrac{d}{dx} x^n$ [$(-1)^{n-1} 3^{n+1} 5$ don't depend on x]

$= \sum_{n=1}^{\infty} (-1)^{n-1} 3^{n+1} 5 n x^{n-1}$.

(b) $\displaystyle\int \sum_{n=0}^{\infty} (-1)^{n-1} 3^{n+1} 5x^n \, dx = \sum_{n=0}^{\infty} \int (-1)^{n-1} 3^{n+1} 5 x^n \, dx$

$= \sum_{n=0}^{\infty} (-1)^{n-1} 3^{n+1} 5 \int x^n \, dx$

$= \sum_{n=0}^{\infty} (-1)^{n-1} 3^{n+1} 5 \dfrac{x^{n+1}}{n+1} + C$

③ $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{n^2}\right|} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n^2}} = \lim_{n\to\infty} \frac{|x|}{(\sqrt[n]{n})^2} = \frac{|x|}{(\lim_{n\to\infty} \sqrt[n]{n})^2} = \frac{|x|}{1^2} = |x|$.

By the Root Test, $\sum_{n=1}^{\infty} \dfrac{x^n}{n^2}$ converges absolutely for $|x|<1$ and diverges for $|x|>1$. Also, $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$ converges absolutely and so does $\sum_{n=1}^{\infty} \dfrac{1^n}{n^2}$.

Then the radius of convergence is $R=1$ and the interval of convergence is $I = [-1,1]$.

Now let's look at its derivative:

$\dfrac{d}{dx} \sum_{n=1}^{\infty} \dfrac{x^n}{n^2} = \sum_{n=1}^{\infty} \dfrac{n x^{n-1}}{n^2} = \sum_{n=1}^{\infty} \dfrac{x^{n-1}}{n}$ $R=1$ (doesn't change)

Note that $\sum \dfrac{(-1)^{n-1}}{n}$ converges (conditionally) but $\sum \dfrac{1^{n-1}}{n}$ diverges.

Then the interval of convergence is $J = [-1,1)$. (it changed!!)

Problems:

① $f(x) = \frac{x^2}{3-2x} = \frac{x^2/3}{1-(2x/3)} = \frac{x^2}{3} \cdot \frac{1}{1-(2x/3)} = \frac{x^2}{3} \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n$

conv. absolutely when $\left|\frac{2x}{3}\right| < 1$

$$\Longleftrightarrow \quad |2x| < 3$$
$$\Longleftrightarrow \quad |x| < 3/2$$
$$\Longleftrightarrow \quad -3/2 < x < 3/2$$

So $R = \frac{3}{2}$

② $\ln(2+t) = \int \frac{1}{2+t}\, dt = \int \frac{1/2}{1-(-t/2)}\, dt = \int \frac{1}{2}\left(\sum_{n=0}^{\infty} \left(\frac{-t}{2}\right)^n\right) dt = C + \frac{1}{\cancel{2}} \sum_{n=0}^{\infty} \frac{-\cancel{2}}{n+1} \cdot \left(\frac{-t}{2}\right)^{n+1}$

We find C by plugging in a value of t. Let's do $t=0$.

$$\Rightarrow \ln(2+0) = C - \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{-0}{2}\right)^{n+1} = C$$

$$\Rightarrow C = \ln(2)$$
$$\Rightarrow \ln(2+t) = \ln(2) - \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \left(-\frac{t}{2}\right)^{n+1}$$

abs. conv. for $\left|\frac{-t}{2}\right| < 1$ (as the radius of convergence doesn't change by differentiation). So $R = 2$.

③ $\arctan(z) = \int \frac{1}{1+z^2}\, dz = \int \frac{1}{1+(-z^2)}\, dz = \int \sum_{n=0}^{\infty} (-z^2)^n\, dz = \sum_{n=0}^{\infty} \int (-z^2)^n\, dz$

$$= \sum_{n=0}^{\infty} (-1)^n \int z^{2n}\, dz = C + \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$$

and $\arctan(0) = 0 = C + \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1} = C \quad \Rightarrow C = 0$

$$\Rightarrow \arctan(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$$

abs. conv. for $|-z^2| < 1$, so $R = 1$.

④ $10x^3 \arctan(2x) = 10x^3 \sum^{\infty} (-1)^n \cdot \frac{(2x)^{2n+1}}{2n+1} = \sum^{\infty} (-10)^n \cdot \frac{2^{2n+1} \cdot x^{2n+4}}{\ldots}$

④ $10x^3\arctan(2x) = 10x^3 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-10)^n \cdot \frac{2^{2n+1} \cdot x^{2n+4}}{2n+1}$

abs. conv. for $|2x| < 1$, so $R = 1/2$.

⑤ $\frac{1}{(2-3x)^2} = \frac{d}{dx}\left(\frac{1/3}{2-3x}\right) = \frac{d}{dx}\left(\frac{1}{6} \cdot \frac{1}{1-(3x/2)}\right) = \frac{1}{6} \cdot \frac{d}{dx}\left(\sum_{n=0}^{\infty} \left(\frac{3x}{2}\right)^n\right) = \frac{1}{6} \cdot \sum_{n=1}^{\infty} \frac{3n}{2} \cdot \left(\frac{3x}{2}\right)^{n-1}$

abs. conv. for $\left|\frac{3x}{2}\right| < 1$, so $R = \frac{2}{3}$.