1. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use.*

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin(n) n^{3/2}}{n^5 + 10} = \sum a_n$$

ABS. CONV. ?

① $|a_n| = \left| (-1)^n \frac{\sin(n)\, n^{3/2}}{n^5 + 10} \right| \leq \frac{n^{3/2}}{n^5+10} \leq \frac{n^{3/2}}{n^5}$ since $|\sin(n)| \leq 1$ and $|(-1)^n| = 1$

② pick $b_n = \frac{n^{3/2}}{n^5} = \frac{1}{n^{3.5}}$. $\sum b_n$ conv by p-test with $p = 3.5 > 1$

③ $|a_n| \leq b_n$ for all $n \Rightarrow \sum |a_n|$ conv as well by p-test

④ So $\sum a_n$ is $\boxed{\text{abs. conv.}}$

2. (10 points) Determine whether the following series converges absolutely, converges only conditionally, or diverges. *Name any test you use.*

ABS. CONV. ?

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n+1}\right)^{n^2} = \sum a_n$$

root test

$$|a_n|^{\frac{1}{n}} = \left|(-1)^n\left(\frac{n}{n+1}\right)^{n^2}\right|^{\frac{1}{n}} = \left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1}$$

$$= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1} \quad \text{since } \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

since $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \frac{1}{e} < 1$, $\sum|a_n|$ conv.

and $\sum a_n$ is $\boxed{\text{abs. conv.}}$ by root test

3. (20 points) Find the radius and interval of convergence of the following power series.

(a)

$$\sum_{n=1}^{\infty} \frac{(-5)^n (x-3)^n}{n4^n}.$$

<u>ratio test</u> $L = \lim_{n\to\infty} \left| \frac{(-5)^{n+1}(x-3)^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{(-5)^n(x-3)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{-5(x-3)}{4} \cdot \underbrace{\cancel{\frac{n}{n+1}}}_{\to 1 \text{ as } n\to\infty} \right| = \frac{5}{4}|x-3|$

So $L < 1$ when $\frac{5}{4}|x-3| < 1$ or $|x-3| < \frac{4}{5}$ or

$-\frac{4}{5} < x-3 < \frac{4}{5}$ $\boxed{R = \frac{4}{5}}$

$-\frac{4}{5} + 3 < x < \frac{4}{5} + 3$

<u>endpoints</u> $x = -\frac{4}{5} + 3 = \frac{11}{5}$:

$\sum \frac{(-5)^n (\frac{11}{5} - 3)^n}{n4^n} = \sum \frac{\cancel{(-5)^n}\cancel{(-4/5)^n}}{n\cancel{4^n}} = \sum \frac{1}{n}$ DIV by p-test

$x = \frac{4}{5} + 3 = \frac{19}{5}$:

$\sum \frac{(-5)^n (\frac{19}{5} - 3)^n}{n4^n} = \sum \frac{(-5)^n (\frac{4}{5})^n}{n4^n} = \sum \frac{(-1)^n}{n}$

CONV by AST

$\boxed{IOC = \left(\frac{11}{5}, \frac{19}{5}\right]}$

note:
$\cos\left(-\frac{3\pi}{2}\right) = 0$
$\sin\left(-\frac{3\pi}{2}\right) = 1$

4. (20 points) Consider the function $f(x) = \cos(3x)$.

(a) Find a power series expansion of $f(x)$ about $x = -\frac{\pi}{2}$.

$f(x) = \cos(3x)$ $\qquad f\left(-\frac{\pi}{2}\right) = 0$ $\qquad C_0 = 0$

$f'(x) = -3\sin(3x)$ $\qquad f'\left(-\frac{\pi}{2}\right) = -3$ $\qquad C_1 = -3$

$f''(x) = -3^2\cos(3x)$ $\qquad f''\left(-\frac{\pi}{2}\right) = 0$ $\qquad C_2 = 0$

$f'''(x) = 3^3\sin(3x)$ $\qquad f'''\left(-\frac{\pi}{2}\right) = 3^3$ $\qquad C_3 = 3^3/3!$

$f^{(4)}(x) = 3^4\cos(3x)$ $\qquad f^{(4)}\left(-\frac{\pi}{2}\right) = 0$ $\qquad C_4 = 0$

$\vdots$ $\qquad C_5 = -3^5/5!$

$$\sum_{n=0}^{\infty} C_n\left(x+\frac{\pi}{2}\right)^n = \boxed{\sum_{k=0}^{\infty} (-1)^{k+1}\frac{3^{2k+1}}{(2k+1)!}\left(x+\frac{\pi}{2}\right)^{2k+1}}$$

$C_{2k} = 0$

$C_{2k+1} = (-1)^{k+1}\, 3^{2k+1}/(2k+1)!$

(b) Use the ratio test to find the radius and interval of convergence of the series you found in (a). *No credit will be given for solutions not using the ratio test.*

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}\,3^{2n+3}}{(2n+3)!}\left(x+\frac{\pi}{2}\right)^{2n+3}\cdot\frac{(2n+1)!}{(-1)^{n+1}\,3^{2n+1}\left(x+\frac{\pi}{2}\right)^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3^2\left(x+\frac{\pi}{2}\right)^2}{(2n+3)(2n+2)}\right| = 0 < 1 \text{ for all } x$$

$R = \infty$

$IOC = (-\infty, \infty)$

5. (20 points)

(a) Find the Maclaurin series expansion of the function

$$f(x) = \frac{x^2 - \arctan(x^2)}{2x^4},$$

write out the first four nonzero terms, and express the series in sigma notation.

$$-\arctan(x^2) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{4n+2} = -x^2 + \frac{x^6}{3} - \frac{x^{10}}{5} + \cdots$$

$$x^2 - \arctan(x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{4n+2} = \frac{x^6}{3} - \frac{x^{10}}{5} + \frac{x^{14}}{7} - \frac{x^{18}}{9} + \cdots$$

$$\frac{x^2 - \arctan(x^2)}{2x^4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2(2n+1)} x^{4n-2} = \frac{x^2}{2\cdot 3} - \frac{x^6}{2\cdot 5} + \frac{x^{10}}{2\cdot 7} - \frac{x^{14}}{2\cdot 9} + \cdots$$

(b) What is the value of $f^{(11)}(0)$?

$f^{(11)}(0) = 0$ since $C_{11} = 0$

(c) What is the value of $f^{(12)}(0)$?

$f^{(12)}(0) = 0$ since $C_{12} = 0$

(d) What is the value of $\lim_{x \to 0} f(x)$?

$$\lim_{x\to 0} f(x) = \lim_{x \to 0} \left(\frac{x^2}{2\cdot 3} - \frac{x^6}{2\cdot 5} + \cdots \right) = 0$$

6. (10 points) Write out the first three terms and then find the sum of each of the following series. *Your table of Maclaurin series expansions might be helpful.*

(a) $\sum_{n=0}^{\infty} \frac{3}{n!}\left(\frac{-1}{2}\right)^n = 3 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-1}{2}\right)^n = 3e^{-1/2}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!} 2^{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} = -\sin(2)$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n6^n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{1}{6}\right)^n = -\ln\left(1+\frac{1}{6}\right) = -\ln\left(\frac{7}{6}\right)$

7. (10 points) Consider the parametric equations for a curve $C(\theta)$ defined by

$$x = 3\tan(\theta), \qquad y = 6\sec(\theta), \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Eliminate the parameter, and write the resulting Cartesian equation in the form given below. *No credit will be given for solutions not showing any work.*

$$\boxed{\frac{y^2}{36} = 1 + \frac{x^2}{9} \qquad y > 0}$$

1. $\frac{x}{3} = \tan\theta \qquad \frac{y}{6} = \sec\theta$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \left(\frac{x}{3}\right)^2 = \left(\frac{y}{6}\right)^2$$

2. $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow \cos\theta > 0 \Rightarrow \sec\theta > 0 \Rightarrow y > 0$