Workshop (8): Absolute Convergence and the Ratio and Root Tests

MTH 143

Warm-up:

1. Find $\lim_{n\to\infty} \sqrt[n]{n}$. (Hint: Start by setting $y = \sqrt[n]{n}$ and taking the natural log of both sides. You will need L'Hopital's rule. The technique you used in this exercise will be needed later in this workshop.)

2. Use the ratio test to determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n3^n}{4^{n+1}}$.

3. Use the root test to determine the convergence of the same series.

Problems:

1. Consider the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

   (a) Apply the ratio test this series. What is the result? Does it depend on p?

   (b) Can you come up with a series $\sum a_n$ such that a_n is a ratio of polynomials in n for which the ratio test provides a conclusive result?

   (c) Now apply the root test to the p-series. Was the test conclusive? (Hint: See warm-up question 1.)

   (d) A generic polynomial can be written in the form

   $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

   Apply the root test to $\sum p(n)$. (Again, you'll need to use the technique from warm-up question 1.) Was the test conclusive? Should you try the ratio test on it?

(e) For which of the following series is the ratio test inconclusive? Try to use ideas from (a)-(d) rather than applying the test. Apply the test if you are unsure.

i. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ ii. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2^n}$ iii. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

2. The terms of a series are defined recursively as follows:

$$a_1 = 2 \qquad a_{n+1} = \frac{5n+1}{4n+3} a_n.$$

Does $\sum a_n$ converge?

3. Determine the conditional convergence, absolute convergence, or divergence of the following series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n}{n10^{n+1}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$

(f) $\displaystyle\sum_{n=1}^{\infty} (\arctan n)^n$