Workshop (7): Alternating Series and Conditional Convergence

MTH 143

Warm-up:

Are the following series alternating?

1. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{-2^n}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{(-2)^n}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{2n+1}}{(-2)^{2n}}$

Problems:

1. Consider the series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}}.$$
   (a) Write out the first 8 terms.
   (b) This is a geometric series–find its sum, S. Plot S on a number line.
   (c) Find the first 3 terms of the sequence $\{S_{2n}\}$, the partial sums whose index is even. Simplify them. Plot these on a the same number line as the sum.
   (d) Do the same for $\{S_{2n-1}\}$.
   (e) Explain why $\{S_{2n}\}$ is increasing and $\{S_{2n-1}\}$ is decreasing. (Hint: Look at the 8 terms you wrote out. Pair them appropriately.)
   (f) Explain why $S_{2n} < S$ for all n and $S_{2n+1} > S$ for all S.
   (g) Find upper and lower bounds for both sequences. Conclude that both are convergent.
   (h) Show that the limit of $\{S_{2n}\}$ and of $\{S_{2n-1}\}$ are equal by noting that
$$S_{2n} = S_{2n-1} + \left(\frac{-1}{2}\right)^{2n}.$$
   (Take the limit of both sides.)

2. (a) Use the alternating series test to show the convergence of

$$\frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \frac{1}{\ln 6} + \frac{1}{\ln 7} - \dots$$

Then find the smallest n such that the error in using S_n as an estimate of the sum is less than one hundredth. (That is, $|R_n| = |S - S_n| < 0.01$.)

(b) Now use the remainder estimate obtained from the alternating series test to find the smallest n such that $|R_n| < 0.01$ for the series

$$\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

(c) Which n is greater? Why?

3. Determine whether each of the following series diverges, converges conditionally, or converges absolutely. Be sure to justify each part of the alternating series test if you use it.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n!}{n^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1} - \sqrt{n})$

4. For what values of p is the series convergent?

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$$