MTH142 Workshop 3: Areas, Riemann Sums, and the Definite Integral

Work on the following problems with your group.

Warm-Up

1. Use the following theorem for parts (a) and (b) of this problem:

Theorem 4: If f is integrable, then $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$.

(a) If $f(x) = \frac{x}{x+3}$, set up, but do not calculate, the limit to find the definite integral of f on $[-1, 1]$.

(b) Calculate $\int_0^2 (3 + 2x)\, dx$ by taking the limit of the Riemann sum. If you do not remember the summation formula for $\sum_{i=1}^{n} i$, look it up.

(c) Use geometry to check your answer to part (b).

Problems

2. With your group, express the limit as a definite integral on the given interval:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8n + 16i}{n^2}\right) \ln\left(1 + \left(\frac{2n + 4i}{n}\right)^2\right)$, $[2, 6]$.

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i^*}{x_i^*}\left(\frac{\pi}{n}\right)$, $[\pi, 2\pi]$, where x_i^* is any sample point in each subinterval.

3. Consider the following integral with your group:

$$\int_0^2 \left(\sqrt{4 - x^2} + 3x\right)\, dx$$

(a) Estimate the integral by using four rectangles of equal width and right endpoints.

(b) Write the integral as the limit of a Riemann sum using right endpoints. (Don't try to *evaluate* the limit — just write down the appropriate limit.)

(c) Evaluate the definite integral by interpreting it in terms of areas. (**Note:** This means use the area interpretation of the integral (not Riemann sums) to find the exact answer. Start by separating the integral so that you can integrate $\sqrt{4 - x^2}$ and $3x$ separately.)

(d) Use a calculator to compare your estimate from part (a) to your answer from part (c).

4. (a) If $f(x) = c$, where c is a positive constant, and constants $a < b$, give a picture of the area under $f(x)$ showing why

$$\int_a^b f(x)\ dx = c(b-a).$$

Note that this is true for any constants a, b, and c, even if $b < a$ or if c is not positive.

(b) Using the fact that if $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$, show the following property:

If $h(x) \geq g(x)$ on the interval $[a, b]$, then

$$\int_a^b h(x)dx \geq \int_a^b g(x)dx.$$

[**Hint:** Since $h(x) \geq g(x)$, then $h(x) - g(x) \geq 0$. Use the function $(h-g)(x)$ in the fact mentioned.]

(c) Using the properties proved in (a) and (b), show the following property:

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$

(d) Use the property in part (c) to estimate the value of $\int_{\pi/4}^{\pi/3} \tan(x)dx$. [**Hint:** $\tan(x)$ is strictly increasing on the interval $[\pi/4, \pi/3]$.]

5. Show that $\int_a^b x dx = \frac{b^2-a^2}{2}$ by taking the limit of the Riemann sum.

Extras

6. Show that $\int_a^b x^2 dx = \frac{b^3-a^3}{3}$ by taking the limit of the Riemann sum.