MTH 142: Calculus II

Final Exam

December 17, 2017

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Indicate your instructor with a check in the appropriate box:

Crossen	MW 9-10:15	☐
Zhong	MW 3:25-4:40	☐

- You have 3 hours to work on this exam.
- You are responsible for checking that this exam has all 18 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.
- Read the following Academic Honesty Statement and sign:

  I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

  Signature:______________________

Part A		
QUESTION	VALUE	SCORE
1	18	
2	10	
3	10	
4	16	
5	21	
TOTAL	75	

Part B		
QUESTION	VALUE	SCORE
1	20	
2	20	
3	10	
4	15	
5	10	
TOTAL	75	

Part A

1. (18 points) Consider the function (used throughout this problem)

$$g(x) = \frac{3x^2 - 3}{(x-3)^2}.$$

(a) Determine the following for $g(x)$.

Domain:

$(x-3)^2 \neq 0$

$x \neq 3$

$\boxed{(-\infty, 3) \cup (3, \infty)}$

y-intercept:

$x = 0 \Rightarrow y = \frac{-3}{(-3)^2} = -1/3$

$\boxed{(0, -1/3)}$

x-intercept(s):

$y = 0 \Rightarrow 0 = 3x^2 - 3$

$0 = 3(x^2 - 1)$

$x = \pm 1$

$\boxed{(\pm 1, 0)}$

Vertical Asymptote(s): possible : $\boxed{x = 3}$

$\lim_{x \to 3} \frac{3x^2-3}{(x-3)^2} = \frac{24}{(0)^2} = +\infty$

Horizontal Asymptote(s):

$\lim_{x \to \infty} \frac{3x^2-3}{(x-3)^2} = 3$

$\lim_{x \to -\infty} \frac{3x^2-3}{(x-3)^2} = 3.$

$\boxed{y = 3}$

(b) For the same $g(x)$, given that

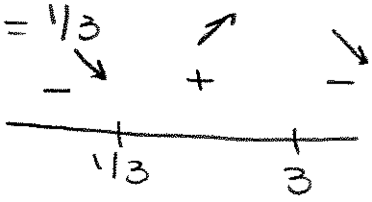
$$g'(x) = \frac{6(1-3x)}{(x-3)^3},$$

determine the following for $g(x)$.

Open intervals on which $g(x)$ is increasing:

$g'(x) = 0 = 1 - 3x \Rightarrow x = 1/3$

g' DNE @ $x = 3$



$\boxed{(1/3, 3)}$

$g'(0) = \frac{+}{-} = -$

$g'(1) = \frac{-}{-} = +$

$g'(100) = \frac{-}{+} = -$

Open intervals on which $g(x)$ is decreasing:

$\boxed{(-\infty, 1/3) \cup (3, \infty)}$

local maximum value(s) of $g(x)$:

$\boxed{\text{none}}$

local minimum value(s) of $g(x)$:

$x = 1/3 \quad g(1/3) = \frac{3(1/3)^2 - 3}{(1/3 - 3)^2} = \frac{\frac{1}{3} - 3}{(\frac{1}{3} - 3)^2}$

$= \frac{1}{1/3 - 3} = \frac{1}{-8/3} = \boxed{-3/8}$

(c) For the same $g(x)$, given that

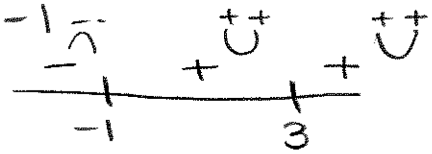
$$g''(x) = \frac{36(x+1)}{(x-3)^4},$$

determine the following for $g(x)$.

Open intervals on which $g(x)$ is concave up:

$g''(x) = 0 = x + 1 \Rightarrow x = -1$

g'' DNE @ $x = 3$



$\boxed{(-1,3) \cup (3,\infty)}$

$g''(-100) = \frac{-}{+}$

$g''(0) = \frac{+}{+}$

$g''(100) = \frac{+}{+}$

Open intervals on which $g(x)$ is concave down:

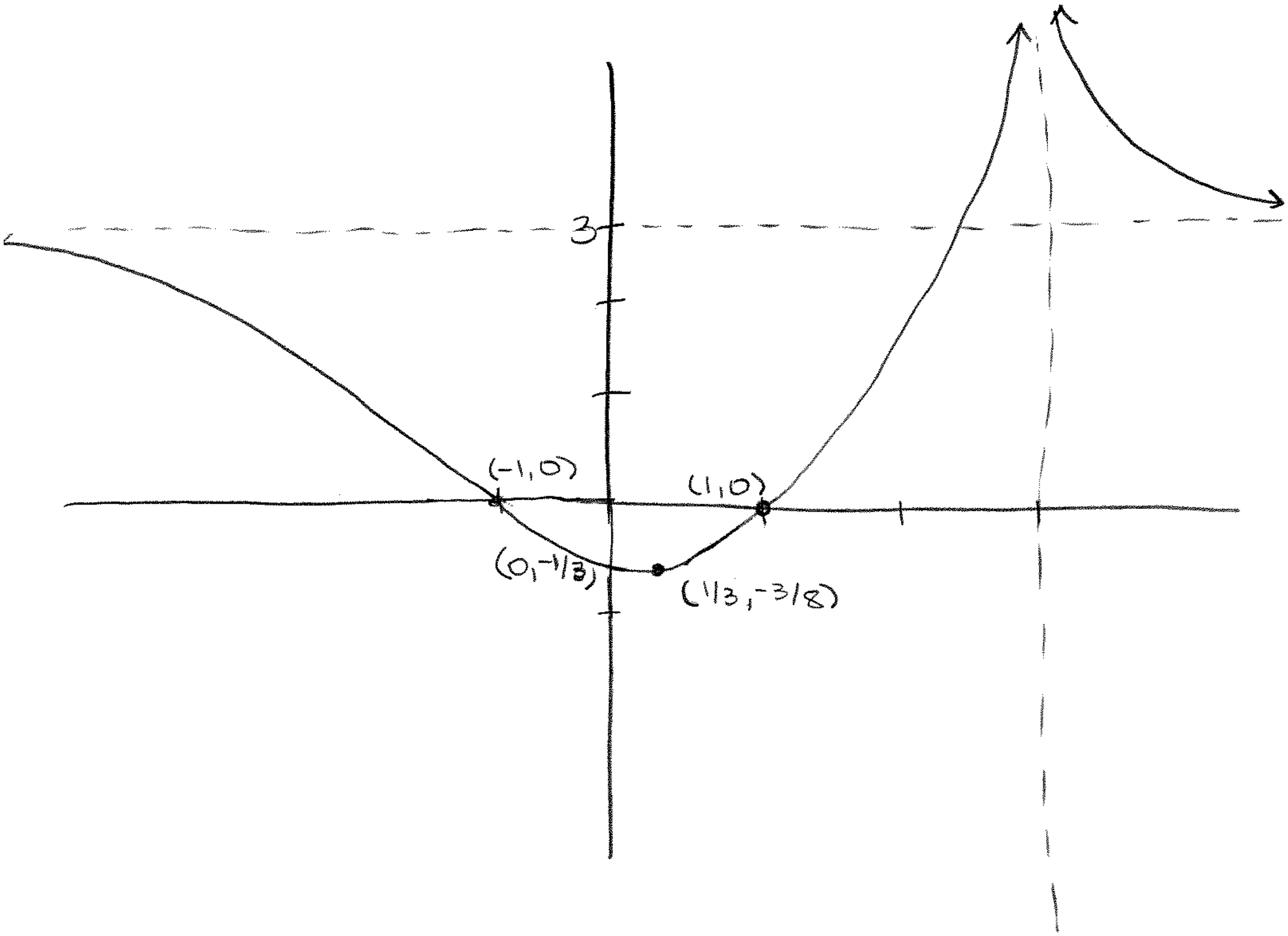
$\boxed{(-\infty,-1)}$

Inflection point(s):

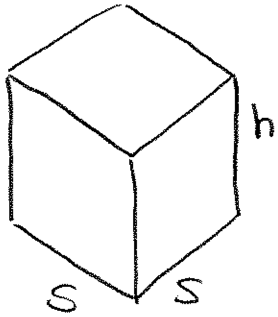
$x = -1 \qquad g(-1) = \frac{3(-1)^2 - 3}{(-1-3)^2} = 0$

$\boxed{(-1, 0)}$.

(d) Sketch the graph of $g(x)$.



2. (10 points) A box with a square base and an open top must have a volume of 1000 cm^3. The cost of the material for the base is \$2/cm^2 and the cost of the material for the sides is \$1/cm^2. Find the dimensions of the box that minimize the cost of materials.



$V = s^2 h = 1000$

$h = 1000 s^{-2}$

$0 < s < \infty$

$- \searrow \quad \nearrow +$

$\overset{}{\underset{10}{|}}$

$C'(1) = -$

$C'(100) = +$

$\Rightarrow s = 10$ is min

$C = (2)(s^2) + (1)(4sh)$

$C = 2s^2 + 4s(1000s^{-2})$

$C = 2s^2 + 4000s^{-1}$

$C' = 4s - 4000s^{-2} = 0$

$4s = \frac{4000}{s^2}$

$4s^3 = 4000$

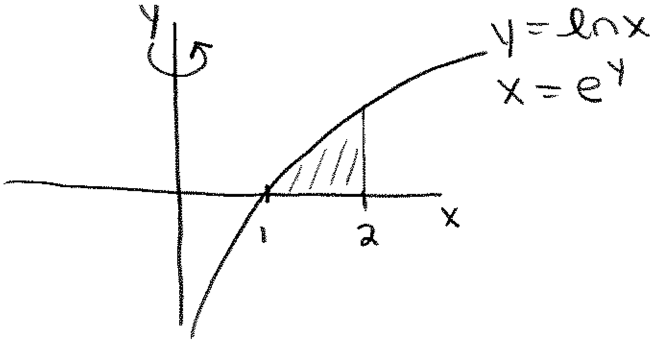
$s^3 = 1000$

$s = 10$ cm.

$h = \frac{1000}{10^2} = 10$

$\boxed{10 \text{ cm} \times 10 \text{ cm} \times 10 \text{ cm}}$

3. (10 points) Calculate the volume generated by rotating the region bounded by the curves $y = \ln x$, $y = 0$, and $x = 2$, about the y-axis. You may use either method of washers/disks or cylindrical shells, but clearly indicate which method you use.



Shells:

$$2\pi\int_1^2 x\ln x\,dx = 2\pi\left[\frac{1}{2}x^2\ln x\Big|_1^2 - \frac{1}{2}\int x\,dx\right]$$

$u = \ln x$
$dv = x\,dx$
$du = \frac{1}{x}dx$
$v = \frac{1}{2}x^2$

$$= 2\pi\left[2\ln(2) - \frac{1}{4}x^2\Big|_1^2\right]$$

$$= 2\pi\left[2\ln(2) - 1 + 1/4\right]$$

$$= 4\pi\ln(2) - 3\pi/2$$

Washers:

$$\pi\int_0^{\ln(2)}\left(2^2 - (e^y)^2\right)dy$$

$$= \pi\int_0^{\ln(2)}(4 - e^{2y})\,dy = \pi\left[4y - \frac{1}{2}e^{2y}\right]_0^{\ln(2)}$$

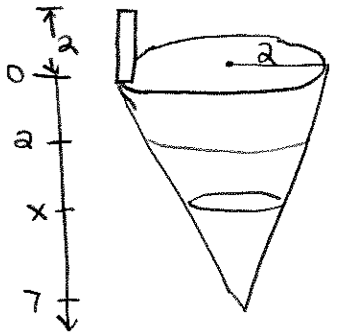
$$= \pi\left[4\ln(2) - \frac{1}{2}e^{2\ln(2)} + 1/2\right]$$

$$= 4\pi\ln(2) - \frac{1}{2}(4)\pi + \pi/2$$

$$= 4\pi\ln(2) - 3\pi/2.$$

4. (16 points) Consider a water tank in the shape of an inverted cone (point down) with a radius of 2 meters, height of 7 meters, and a spout extending 2 meters above the top of the tank. (Recall the gravitational constant is $g = 9.8$ m/s^2 and the density of water is 1000 kg/m^3)

(a) If the tank is only filled to a height of 5 meters with water, find an integral that represents the total work required to pump all of the water out of the spout. **(Do not evaluate the integral.)**

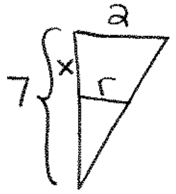


$F_{slice} = (9.8\ m/s^2)(1000\ kg/m^3)\ V_{slice}$

$V_{slice} = \pi\, r(x)^2\, \Delta x\ \ m^3$

$r(x) = \frac{2}{7}(7-x)\ \ m$

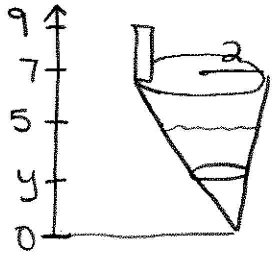
$d_{slice} = x + 2\ \ m$



$\frac{2}{r} = \frac{7}{7-x}$

$r = \frac{2}{7}(7-x)$

$W = \int_2^7 (9.8)(1000)\pi\left(\frac{2}{7}(7-x)\right)^2 (x+2)\,dx$ J.

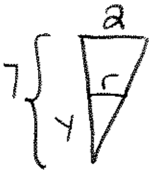


$F_{slice} = (9.8\ m/s^2)(1000\ kg/m^3)\ V_{slice}$

$V_{slice} = \pi\, r(y)^2\, \Delta y\ \ m^3$

$r(y) = \frac{2}{7} y\ \ m$

$d_{slice} = 9 - y$



$\frac{2}{r} = \frac{7}{y}$

$r = \frac{2}{7} y$

$W = \int_0^5 (9.8)(1000)\pi\left(\frac{2}{7}y\right)^2 (9-y)\,dy$ J.

(b) If the tank is instead completely filled with water, find an integral that represents the total work required to pump water out of the spout until **ONLY 2 METERS OF WATER REMAIN IN THE TANK. (Do not evaluate the integral.)**

$$W = \int_0^5 (9.8)(1000)\pi\left(\frac{2}{7}(7-x)\right)^2 (x+2)\,dx \text{ J}$$

$$W = \int_2^7 (9.8)(1000)\pi\left(\frac{2}{7}y\right)^2 (9-y)\,dy \text{ J}$$

5. (21 points) Evaluate the following integrals.

(a) $\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

$u = 1+\sqrt{x} \Rightarrow \sqrt{x} = u-1$

$du = \frac{1}{2}x^{-1/2}dx$

$2\sqrt{x}\,du = dx$

$2(u-1)du = dx$

$u(0)=1$

$u(1)=2$

$$\int_1^2 \frac{2(u-1)}{u^4}du = 2\int_1^2 (u^{-3}-u^{-4})du$$

$$= 2\left[\frac{-1}{2}u^{-2}+\frac{1}{3}u^{-3}\right]_1^2$$

$$= 2\left[-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{8}\right)\right]-2\left[-1/2+1/3\right]$$

(b) $\int \tan^3 x \sec^4 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$\int \tan^3 x(\tan^2 x+1)\sec^2 x\,dx$

$= \int u^3(u^2+1)du$

$= \int (u^5+u^3)du$

$= \frac{1}{6}u^6+\frac{1}{4}u^4 + C$

$= \frac{1}{6}\tan^6 x+\frac{1}{4}\tan^4 x + C$

$u = \sec x$

$du = \sec x\tan x\,dx$

$\int (\sec^2 x-1)\sec^3 x\cdot\sec x\tan x\,dx$

$= \int (u^2-1)u^3 du$

$= \int (u^5-u^3)du$

$= \frac{1}{6}u^6-\frac{1}{4}u^4 + C$

$= \frac{1}{6}\sec^6 x-\frac{1}{4}\sec^4 x + C$

(c) $\int e^{2y} \cos y \, dy$

$u = \cos y$

$dv = e^{2y} dy$

$du = -\sin y \, dy$

$v = \frac{1}{2} e^{2y}$

$$= \frac{1}{2} e^{2y} \cos y + \frac{1}{2} \int e^{2y} \sin y \, dy$$

$u = \sin y$

$dv = e^{2y} dy$

$du = \cos y \, dy$

$v = \frac{1}{2} e^{2y}$

$$= \frac{1}{2} e^{2y} \cos y + \frac{1}{2} \left[\frac{1}{2} e^{2y} \sin y - \frac{1}{2} \int e^{2y} \cos y \, dy \right]$$

$$\int e^{2y} \cos y \, dy = \frac{1}{2} e^{2y} \cos y + \frac{1}{4} e^{2y} \sin y - \frac{1}{4} \int e^{2y} \cos y \, dy$$

$$\frac{5}{4} \int e^{2y} \cos y \, dy = \frac{1}{2} e^{2y} \cos y + \frac{1}{4} e^{2y} \sin y$$

$$\int e^{2y} \cos y \, dy = \frac{4}{5} \left[\frac{1}{2} e^{2y} \cos y + \frac{1}{4} e^{2y} \sin y \right] + C$$

$u = e^{2y}$

$dv = \cos y \, dy$

$du = 2e^{2y} dy$

$v = \sin y$

$$= e^{2y} \sin y - 2 \int e^{2y} \sin y \, dy$$

$u = e^{2y}$

$dv = \sin y \, dy$

$du = 2e^{2y} dy$

$v = -\cos y$

$$= e^{2y} \sin y - 2 \left[-e^{2y} \cos y + 2 \int e^{2y} \cos y \, dy \right]$$

$$\int e^{2y} \cos y \, dy = e^{2y} \sin y + 2e^{2y} \cos y - 4 \int e^{2y} \cos y \, dy$$

$$5 \int e^{2y} \cos y \, dy = e^{2y} \sin y + 2e^{2y} \cos y$$

$$\int e^{2y} \cos y \, dy = \frac{1}{5} \left[e^{2y} \sin y + 2e^{2y} \cos y \right] + C$$

Part B

1. (20 points) Evaluate the following integrals:

(a) $\int \frac{\sqrt{x^2-9}}{2x}\,dx$

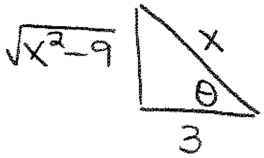
$$= \int \frac{\sqrt{9\sec^2\theta - 9}\,(3\sec\theta\tan\theta)d\theta}{2\cdot 3\sec\theta}$$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta d\theta$

$$= \int \frac{(3\tan\theta)(\cancel{3\sec\theta})\tan\theta\, d\theta}{2(\cancel{3\sec\theta})}$$

$\sec\theta = x/3 = H/A$



$$= \frac{3}{2}\int \tan^2\theta\, d\theta = \frac{3}{2}\int(\sec^2\theta - 1)d\theta$$

$$= \frac{3}{2}\left[\tan\theta - \theta\right] + C$$

$$= \frac{3}{2}\left[\frac{\sqrt{x^2-9}}{3} - \arccos(3/x)\right] + C$$

(b) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

$$= \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta d\theta$

$$= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\,(2\sec\theta)}$$

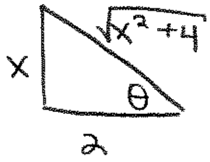
$u = \sin\theta$

$du = \cos\theta d\theta$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$\tan\theta = \frac{x}{2} = \frac{O}{A}$

$$= \frac{1}{4}\int\left(\frac{1}{\cos\theta}\right)\left(\frac{\cos^2\theta}{\sin^2\theta}\right)d\theta = \frac{1}{4}\int \frac{\cos\theta d\theta}{\sin^2\theta}$$



$$= \frac{1}{4}\int \frac{du}{u^2} = \frac{-1}{4u} + C = \frac{-1}{4\sin\theta} + C$$

$$= \frac{-\sqrt{x^2+4}}{4x} + C$$

2. (20 points) Evaluate the following integrals:

(a) $\int \frac{x^2-1}{x^3+x}\,dx.$ $= \int \frac{x^2-1}{x(x^2+1)}dx = \int\left[\frac{A}{x} + \frac{Bx+C}{x^2+1}\right]dx$

$$x^2-1 = A(x^2+1)+(Bx+C)x$$

$$x^2-1 = Ax^2+A+Bx^2+Cx$$

$$A=-1\ ,\ C=0$$

$$1=A+B \Rightarrow B=2$$

$$= \int -\frac{1}{x}dx + \int \frac{2x\,dx}{x^2+1}$$

$$= -\ln|x| + \int \frac{du}{u} \qquad u=x^2+1 \quad du=2x\,dx$$

$$= -\ln|x| + \ln|u| + C$$

$$= -\ln|x| + \ln|x^2+1| + C$$

$$= \ln\left|\frac{x^2+1}{x}\right| + C$$

(b) $\int \frac{x^3-3x^2-x-1}{x^2-2x-3}\,dx$ $= \int\left[x-1+\frac{-4}{(x-3)(x+1)}\right]dx$

$$\begin{array}{r} x-1+\frac{-4}{x^2-2x-3} \\ x^2-2x-3 \overline{\big)\, x^3-3x^2-x-1} \\ \underline{-(x^3-2x^2-3x)} \\ -x^2+2x-1 \\ \underline{-(-x^2+2x+3)} \\ -4 \end{array}$$

$$-4 = A(x+1)+B(x-3)$$

$$x=-1 \Rightarrow -4 = B(-4) \Rightarrow B=1$$

$$x=3 \Rightarrow -4 = A(4) \Rightarrow A=-1$$

$$= \int\left[x-1+\frac{A}{x-3}+\frac{B}{x+1}\right]dx$$

$$= \int\left[x-1+\frac{-1}{x-3}+\frac{1}{x+1}\right]dx$$

$$= \frac{1}{2}x^2 - x - \ln|x-3| + \ln|x+1| + C.$$

3. (10 points) Set up the partial fraction decomposition for the following integral in terms of variables but do not solve for those variables.

$$\int \frac{1}{(x^3+x^2+x)(x^3-x)(x^2+x)(x^2+x+1)}\,dx$$

$$= \int \frac{dx}{x(x^2+x+1)x(x-1)(x+1)x(x+1)(x^2+x+1)}$$

$$= \int \frac{dx}{x^3(x-1)(x+1)^2(x^2+x+1)^2}$$

$$= \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{x+1} + \frac{F}{(x+1)^2} + \frac{Gx+H}{x^2+x+1} + \frac{Ix+J}{(x^2+x+1)^2} \right] dx$$

4. (15 points) Determine if each integral is convergent or divergent. Evaluate those that are convergent. Show all your work.

(a) $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_0^{\ln(t)} u\,du$

$u=\ln x$
$du=\frac{1}{x}dx$
$u(1)=0$
$u(t)=\ln(t)$

$= \lim_{t\to\infty} \frac{1}{2}u^2\Big|_0^{\ln(t)}$

$= \lim_{t\to\infty} \frac{1}{2}(\ln(t))^2 \to \infty$.

(b) $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{dx}{\sqrt[3]{x-1}} + \int_1^9 \frac{dx}{\sqrt[3]{x-1}}$

$\int_0^1 (x-1)^{-1/3}dx = \lim_{t\to 1^-}\int_0^t (x-1)^{-1/3}dx = \lim_{t\to 1^-} \frac{3}{2}(x-1)^{2/3}\Big|_0^t$

$= \lim_{t\to 1^-} \frac{3}{2}(t-1)^{2/3} - \frac{3}{2} = -3/2$.

$\int_1^9 (x-1)^{-1/3}dx = \lim_{t\to 1^+}\int_t^9 (x-1)^{-1/3}dx = \lim_{t\to 1^+} \frac{3}{2}(x-1)^{2/3}\Big|_t^9$

$= \lim_{t\to 1^+} \frac{3}{2}(8)^{2/3} - \frac{3}{2}(t-1)^{2/3} = \frac{3}{2}(2)^2 = 6$

$\Rightarrow \int_0^9 \frac{dx}{\sqrt[3]{x-1}} = -\frac{3}{2} + 6 = 9/2$.

5. (10 points) Find the arc length of the curve

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 2.$$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

$$\left(\frac{dy}{dx}\right)^2 = x^4 - 1/2 + \frac{1}{16x^4}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = x^4 + 1/2 + \frac{1}{16x^4}$$

$$= \left(x^2 + \frac{1}{4x}\right)^2$$

$$\downarrow$$

$$L = \int_1^2 \left(x^2 + \frac{1}{4x}\right) dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4}\ln|x|\right]_1^2$$

$$= \boxed{\frac{1}{3}(8) - \frac{1}{4}\ln(2) - \frac{1}{3}}$$

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Formula Sheet

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin(2x) = 2 \sin x \cos x$
- $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$
- $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\sin x \cos y = \dfrac{\sin(x - y) + \sin(x + y)}{2}$
- $\sin x \sin y = \dfrac{\cos(x - y) - \cos(x + y)}{2}$
- $\cos x \cos y = \dfrac{\cos(x - y) + \cos(x + y)}{2}$