MATH 142

Final ANSWERS

May 6, 2014

Part A

1. (24 points)

(a), (6 points) Find the vertical and horizontal asymptotes of

$$f(x) = \frac{3x^2 + 2x + 4}{2x^2 + x - 1}.$$

Answer:

(a) The denominator factors into $(2x-1)(x+1)$. Therefore the function has vertical asymptotes at $x = \frac{1}{2}$ and $x = -1$.

To find the horizontal asymptotes, we let $x \to \infty$ and $x \to -\infty$. Factoring out the leading term x^2 from both numerator and denominator, we find

$$\lim_{x\to\infty} \frac{3x^2 + 2x + 4}{2x^2 + x - 1} = \lim_{x\to\infty} \frac{x^2}{x^2} \cdot \frac{3 + (2/x) + (4/x^2)}{2 + (1/x) - (1/x^2)} = \frac{3}{2}$$

So $y = \frac{3}{2}$ is a horizontal asymptote as $x \to \infty$. A similar calculation shows that $y = \frac{3}{2}$ is a horizontal asymptote as $x \to -\infty$.

(b), (6 points) Does the following function have any symmetry? If so, what kind of symmetry does it have?

$$f(x) = \frac{x^3 \sin(x)}{2 + x^2} - e^{x^2}$$

Answer:

(b) Plugging in $-x$, we find that

$$f(-x) = \frac{(-x)^3 \sin(-x)}{2 + (-x)^2} - e^{(-x)^2}$$

Note that $\sin(x)$ and x^3 are odd functions, but x^2 is even. Therefore,

$$f(-x) = \frac{(-1)x^3(-1)\sin(x)}{2 + x^2} - e^{x^2} = \frac{\sin(x)x^3}{2 + x^2} - e^{x^2} = f(x)$$

and therefore the function is symmetric with an even symmetry.

(c), (6 points) Find the intervals of increase and decrease for the following function. Then find the points x where the function has a local maximum or local minimum.

$$f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 2x + 5$$

Answer:

(c) Note that

$$f'(x) = 2x^2 - 3x - 2 = (2x+1)(x-2)$$

The critical points are at $x = -\frac{1}{2}, 2$. These divide the x-axis into three intervals: $(-\infty, -\frac{1}{2})$, $(-\frac{1}{2}, 2)$ and $(2, \infty)$. Checking the sign of $f'(x)$ at representative points on these intervals, we find that $(-\infty, -\frac{1}{2})$ and $(2, \infty)$ are intervals of increase, while $(-\frac{1}{2}, 2)$ is an interval of decrease. The first derivative test then tells us that $f(x)$ has a local maximum at $x = -\frac{1}{2}$ and a local minimum at $x = 2$.

(d), (6 points) Using the same function $f(x)$ as in part (c), find the intervals on which the function is concave up and concave down, and find the points of inflection.

Answer:

(d) Note that

$$f''(x) = 4x - 3$$

So $f''(x) = 0$ at $x = \frac{3}{4}$. This point divides the x-axis into the intervals $(-\infty, \frac{3}{4})$ and $(\frac{3}{4}, \infty)$. Checking the sign of $f''(x)$ at representative points on these intervals, we find that $f''(x) > 0$ on $(\frac{3}{4}, \infty)$ so f is concave up on this interval. Also, $f''(x) < 0$ on $(-\infty, \frac{3}{4})$ so f is concave down on this interval. Since the concavity changes sign at $x = \frac{3}{4}$ then it is a point of inflection.

2. (15 points) A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 9 - x^2$ above the x-axis. What are the dimensions of such a rectangle with the greatest possible area?

Answer:

Let x be the x-coordinate of the right side of a rectangle, so $0 \le x \le 3$. The width is $2x$ and the height is $9 - x^2$ and thus the area is $A(x) = 2x(9 - x^2)$. To find the maximum area, solve $A'(x) = 18 - 6x^2 = 0$, then $x = \sqrt{3}$. Since $A''(\sqrt{3}) = -12\sqrt{3} < 0$, $A(x)$ has its maximum at $x = \sqrt{3}$. Thus, the width is $2\sqrt{3}$ and the height is 6.

3. (19 points)

Evaluate the following integrals.

(a) (6 points)

$$\int x \sec^2(x^2+1)dx$$

Answer:

We use the substitution $u = x^2 + 1$, $du = 2xdx$. Then

$$\begin{aligned}\int x \sec^2(x^2+1)dx &= \frac{1}{2}\int \sec^2(u)du \\ &= \frac{1}{2}\tan(u) + C \\ &= \frac{1}{2}\tan(x^2+1) + C.\end{aligned}$$

(b) (7 points)

$$\int (x+1)^{555} x^2 dx$$

Answer:

We use the substitution $u = x + 1$, $du = dx$. Since $x = u - 1$, we have

$$\begin{aligned}\int (x+1)^{555} x^2 dx &= \int u^{555}(u-1)^2 du \\ &= \int u^{555}(u^2 - 2u + 1)du \\ &= \int (u^{557} - 2u^{556} + u^{555})du \\ &= \frac{u^{558}}{558} - 2\frac{u^{557}}{557} + \frac{u^{556}}{556} + C \\ &= \frac{(x+1)^{558}}{558} - 2\frac{(x+1)^{557}}{557} + \frac{(x+1)^{556}}{556} + C.\end{aligned}$$

(c) (6 points)

$$\int_0^{\pi} \sin(x)e^{\cos(x)}dx$$

Answer:

We use the substitution $u = \cos(x)$, $du = -\sin(x)dx$.

$$\begin{aligned}\int_{x=0}^{x=\pi} \sin(x)e^{\cos(x)}dx &= -\int_{u=1}^{u=-1} e^u du \\ &= -e^u\Big|_{u=1}^{u=-1} \\ &= -e^{-1} + e^1\end{aligned}$$

4. (12 points) Find the area enclosed by the curves $y = 6x^2$ and $y = 2x^3$

Answer:

(a) To find the points of intersection, set these functions equal and solve: $6x^2 = 2x^3$ gives $x = 0, 3$ as solutions. On $(0, 3)$, $6x^2 > 2x^3$ (one can see this by observing that $6x^2 - 2x^3 = 2x^2(3-x)$, which is positive on the indicated region), so the integral representing the area is given by

$$\int_0^3 \left(6x^2 - 2x^3\right) dx = \left[2x^3 - \frac{1}{2}x^4\right]\Big|_0^3 = 54 - \frac{81}{2} = \frac{27}{2}.$$

5. (15 points) Find the volume of the solid obtained by rotating the region in the first quadrant bounded by $y = x^3$, $y = x$ about the line $y = 4$ using the method of discs or washers.

Answer:

$$\begin{aligned}V &= \int_0^1 \left(\pi(4-x^3)^2 - \pi(4-x)^2\right) dx \\ &= \pi\int_0^1 ((16 - 8x^3 + x^6) - (16 - 8x + x^2))dx \\ &= \pi\int_0^1 (x^6 - 8x^3 - x^2 + 8x)dx \\ &= \pi\left[\left(\frac{1}{7}x^7 - 2x^4 - \frac{1}{3}x^3 + 4x^2\right)\right]_0^1 \\ &= \frac{38\pi}{21}\end{aligned}$$

6. (15 points) Find the volume of the solid obtained by rotating the region bounded by $y = 1 - x^2$, $y = 0$ about the line $x = 2$ using cylindrical shells.

Answer:

$$
\begin{aligned}
V &= \int_{-1}^{1} 2\pi(2-x)(1-x^2)dx \\
&= \int_{-1}^{1} 2\pi(2-x-2x^2+x^3)dx \\
&= \left[2\pi\left(2x-\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4\right)\right]_{-1}^{1} \\
&= \frac{16\pi}{3}
\end{aligned}
$$

Part B

1. (19 points) The great pyramid of Giza is approximately 140 meters high, and the base is a square of side length approximately 440 meters. Suppose that Egypt's ruling generals decide to hollow out the pyramid and use it as a water tower. Compute the work required to pump water from the base of the pyramid until it completely fills the space available.

Since you are not allowed to have a calculator, you do not have to completely simplify your answer.

Hint: Let x be the distance from the top of the pyramid. A horizontal slice of the pyramid at position x is a square, and by using some geometry you can compute the side length of this square as a function of x.

Answer:

Define x as in the hint, and recall that the density of water is 1000kg per cubic meter. Since we are working in the metric system, we must use the gravitational constant $g = 9.8$.

Let $L = L(x)$ be the side length of a square slice at distance x below the top. Using similar triangles, we compute

$$\frac{L}{x} = \frac{440}{140}, \qquad L = \frac{440}{140}x.$$

and so the volume of the slice is

$$L^2 dx = \left(\frac{440}{140}\right)^2 x^2 dx$$

The mass of the slice is

$$1000\left(\frac{440}{140}\right)^2 x^2 dx$$

and the force on the slice is the mass multiplied by 9.8, giving

$$9800\left(\frac{440}{140}\right)^2 x^2dx$$

The distance of the slice from the bottom is $140 - x$, so the work required to fill the slice is

$$9800\left(\frac{440}{140}\right)^2 x^2(140-x)dx = 9800\left(\frac{440}{140}\right)^2 (140x^2 - x^3)dx$$

Finally, the total work required, in Joules, is

$$\begin{aligned}\int_0^{140} 9800\left(\frac{440}{140}\right)^2 (140x^2 - x^3)dx &= 9800\left(\frac{440}{140}\right)^2 \left(\frac{140x^3}{3} - \frac{x^4}{4}\right)\Big|_0^{140} \\ &= 9800\left(\frac{440}{140}\right)^2 \left(\frac{140^4}{3} - \frac{140^4}{4}\right) \\ &= 9800 \cdot 440^2 \cdot \frac{140^2}{12}\end{aligned}$$

2. (22 points) (a) (11 points) Evaluate

$$\int_0^{\pi} x\sin(2x)dx$$

Answer:

We use integration by parts, setting $u = x$, $dv = \sin(2x)dx$. Then $du = dx$, $v = \frac{-1}{2}\cos(2x)$, and we have

$$\begin{aligned}\int_0^{\pi} x\sin(2x)dx &= \left[\frac{-x}{2}\cos(2x)\right]_0^{\pi} - \int_0^{\pi}\frac{-1}{2}\cos(2x)dx \\ &= \frac{-\pi}{2} + \frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^{\pi} \\ &= \frac{-\pi}{2}.\end{aligned}$$

(b) (11 points) Evaluate

$$\int x^2e^{3x}dx$$

Answer:

We use integration by parts, setting $u = x^2$, $dv = e^{3x}$. Then $du = 2xdx$, $v = \frac{1}{3}e^{3x}$, and we have

$$\int x^2e^{3x}dx = \frac{x^2}{3}e^{3x} - \frac{2}{3}\int xe^{3x}dx$$

We now apply integration by parts again for $\int xe^{3x}dx$, setting $u = x$ and $dv = e^{3x}dx$. Then $du = dx$ and $v = \frac{1}{3}e^{3x}$, and we have

$$\int xe^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}$$

Plugging this into the previous, we have

$$\int x^2e^{3x}dx = \frac{x^2}{3}e^{3x} - \frac{2x}{9}e^{3x} + \frac{2}{27}e^{3x} + C.$$

3. (13 points) Evaluate the integral

$$\int \sec^4(\theta)d\theta.$$

Answer:

$$\int \sec^4(\theta)d\theta = \int \sec^2(\theta)\cdot\sec^2(\theta)d\theta = \int (1+\tan^2(\theta))\sec^2(\theta)d\theta$$

We then use the substitution $u = \tan(\theta)$, $du = \sec^2(\theta)d\theta$ and obtain

$$= \int (1+u^2)du = u + \frac{1}{3}u^3 + C = \tan(\theta) + \frac{1}{3}\tan^3(\theta) + C.$$

4. (13 points) Evaluate the integral

$$\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}}dx.$$

Answer:

We first use the trig. sub. $x = \sin\theta$, $dx = \cos\theta d\theta$ and $\sqrt{1-x^2} = \cos\theta$. Then

$$\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}}dx = \int_0^{\pi/3} \frac{\sin^3\theta}{\cos\theta}\cos\theta d\theta = \int_0^{\pi/3} \sin^3\theta d\theta.$$

By another substitution $u = \cos\theta$, $du = -\sin\theta d\theta$ and $\sin^2\theta = 1-\cos^2\theta = 1-u^2$, we obtain

$$= -\int_1^{1/2}(1-u^2)du = \Big[-u + \frac{1}{3}u^3\Big]_1^{1/2} = 5/24.$$

5. (22 points)

Evaluate the following integrals.

(a) (11 points)

$$\int \frac{1}{2x^2 - 7x - 15} dx$$

Answer:

First we factor $2x^2 - 7x - 15 = (2x + 3)(x - 5)$. We must find A and B such that

$$\frac{1}{(2x+3)(x-5)} = \frac{A}{2x+3} + \frac{B}{x-5}$$

Finding a common denominator, we compute

$$\frac{1}{(2x+3)(x-5)} = \frac{A(x-5) + B(2x+3)}{(2x+3)(x-5)}$$

and so

$$1 = A(x-5) + B(2x+3)$$

Setting $x = 5$ we find $1 = 13B$, so $B = 1/13$. Setting $x = -3/2$, we get $1 = -(13/2)A$ and so $A = -2/13$. Thus,

$$\begin{aligned} \int \frac{1}{2x^2 - 7x - 15} dx &= \int \frac{1}{(2x+3)(x-5)} dx \\ &= -\frac{2}{13} \int \frac{1}{2x+3} dx + \frac{1}{13} \int \frac{1}{x-5} dx \\ &= -\frac{2}{13} \cdot \frac{1}{2} \ln|2x+3| + \frac{1}{13} \ln|x-5| + C \\ &= -\frac{1}{13} \ln|2x+3| + \frac{1}{13} \ln|x-5| + C \end{aligned}$$

(b) (11 points)

$$\int \frac{x^2 + 2x + 2}{x - 2} dx$$

Answer:

Since the degree of the numerator is greater than or equal to the degree of the denominator, we must do long division of polynomials. This gives

$$\frac{x^2 + 2x + 2}{x - 2} = x + 4 + \frac{10}{x - 2}$$

and so

$$\int \frac{x^2+2x+2}{x-2}dx = \int (x+4)dx + 10\int \frac{1}{x-2}dx$$
$$= \frac{x^2}{2} + 4x + 10\ln|x-2| + C$$

6. (11 points) Find the arc length of the curve $y = 2(x-1)^{\frac{3}{2}}$ for $1 \le x \le 3$.

Answer:

$y' = \frac{3}{2} \cdot 2\sqrt{x-1} = 3\sqrt{x-1}$, so the arc length is given by

$$\int_1^3 \sqrt{1+9(x-1)}dx = \int_1^3 (9x-8)^{1/2}dx$$
$$= \left[\frac{2}{3} \cdot \frac{1}{9}(9x-8)^{\frac{3}{2}}\right]\Big|_1^3$$
$$= \frac{2}{27}\left(19^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$
$$= \frac{2}{27}\left(19^{\frac{3}{2}} - 1\right).$$