MATH 142

MIDTERM EXAM II

April 3, 2003

1. (12 pts) Find $F'(x)$ for F as given:

(a) $F(x) = \int_{-2}^{x} \sqrt{t^2 - 2t + 5}\, dt$

$$F(x)' = \sqrt{x^2 - 2x + 5}$$

(b) $F(x) = \int_{0}^{x^3} \sec t\, dt$

$$F(x) = 3x^2 \sec(x^3)$$

2. (12 pts) Evaluate the following integrals.

(a) $\int 1\, dx = x + C$

(b) $\int_{-\pi/2}^{\pi/2} \sin^7 x\, dx = 0$
since $\sin^7 x$ is an odd function.

(c) $\int \tan x\, dx$

Use substitution: $u = \cos x$, and $du = -\sin x dx$. Then

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{du}{u} = -\ln|u| + c = -\ln|\cos x| + C$$

3. (16 pts) Find the area of the region(s) bounded by the given functions:

(a) $f(x) = x^2 - 4x + 3$
$g(x) = -x^2 + 2x + 3.$

Find the intersection points of these functions.

$$x^2 - 4x + 3 = -x^2 + 2x + 3$$

$$2x^2 - 6x = 0$$

$$x = 3, \ x = 0$$

On $[0, \ 3]$, $g(x) \geq f(x)$ (test it by plugging in a point). Therefore the area is

$$\int_0^3 (g(x) - f(x))dx = \int_0^3 (-2x^2 + 6x)dx = 9$$

(b) $y = x^3 - 2x$
$y = 2x.$

For the intersection points, solve:

$$x^3 - 2x = 2x$$

$$x^3 - 4x = x(x^2 - 4) = 0$$

so $x = -2, \ 0, \ 2$, and there are two intervals to consider (the region is composed of two pieces).

On $[-2, 0]$, $x^3 - 2x \geq 2x$, while on $[0, \ 2]$, $2x \geq (x^3 - 2x)$ (plug in test points to check).

The area is

$$\int_{-2}^0 [(x^3 - 2x) - 2x]dx + \int_0^2 (2x - (x^3 - 2x))dx$$
$$= 4 + 4 = 8.$$

4. (10 pts) Find the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$ and the line $y = x$ around the horizontal line $y = -1$.

These two curves intersect at 0 and 1, and in $[0,\ 1]$, $x \geq x^2$.
The region has a nice top and bottom, so we can use dx. This will give washers. The cross-sectional area at x is

$$A(x) = \pi(x+1)^2 - \pi(x^2+1)^2.$$

The volume is

$$V = \pi \int_0^1 ((x+1)^2 - (x^2+1)^2)dx = \pi \int_0^1 (2x - x^4 - x^2)dx = \frac{7\pi}{15}$$

5. (10 pts) Find the volume of the solid obtained by rotating the region bounded by the following four lines:

the x-axis, $\quad y = x, \quad y = x - 2, \quad$ and the horizontal line $y = 1$,

around the x-axis.

The top boundary of the region is formed by two different lines. Therefore dx is not ideal. The sides of the region are nice single lines, so dy is a better choice. This will give cylindrical shells.
The circumference of each shell is $2\pi(y)$, and the height is $(y+2) - y = 2$. The volume is

$$V = \int_0^1 2\pi(y)2dy = 4\pi \int_0^1 y\ dy = 2\pi$$

6. (12 pts) A cylindrical well is 12 feet deep with a radius of 3 feet. The well contains 9 feet of water, measured from the bottom. How much work is required to pump all of the water up to ground level?
(Recall that water weighs 62.5 lbs/ft^3.)

Divide $[0,\ 12]$ into n subintervals with the same width Δx. Take a slice at x_i. Then the volume of this slice is $V_i = 9\pi\Delta x$. Hence

$$F_i = 62.5V_i = 62.5 * 9\pi\Delta x$$

is the weight of the slice, and

$$W_i \approx F_i(12 - x_i) = 62.5 * 9\pi(12 - x_i)\Delta x.$$

If we add up all bits of work and let $n \to \infty$, then we obtain that

$$W = \int_0^9 62.5 * 9\pi(12 - x)dx = 62.5 * 9\pi(12x - \frac{x^2}{2})|_0^9 = 62.5 * 9\pi * 57.5$$

7. (12 pts) The wavelength of light emitted by supernova at time t is

$$w(t) = \frac{t^2 + 1}{t^2} \text{ nanometers.}$$

Find the average wavelength between $t = 1/2$ and $t = 2$.

$$w_{ave}(t) = \frac{1}{2 - \frac{1}{2}} \int_{1/2}^{2} \frac{t^2 + 1}{t^2} dt = \frac{2}{3} \int_{1/2}^{2} (1 + \frac{1}{t^2})dt = \frac{2}{3}(t - t^{-1})\Big|_{1/2}^{2} = 2$$

8. (8 pts) Evaluate the following integrals.

(a) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

Use substitution: $u = 1 - x^2$ and $du = -2xdx$. When $x = 0$, $u = 1$, and when $x = 1$, $u = 0$. Thus

$$\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{2}\int_1^0 \sqrt{u}du = \frac{1}{3}$$

(b) $\displaystyle\int t\sqrt{t-4}\,dt$

Use substitution: $u = t - 4$ and $du = dt$. Then

$$\int t\sqrt{t-t}dt = \int (u+4)\sqrt{u}\,du = \int (u^{3/2} + 4u^{1/2})\,du$$

$$= \frac{2}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C = \frac{2}{5}(t-4)^{5/2} + \frac{8}{3}(t-4)^{3/2} + C$$

9. (8 pts) Evaluate the following integrals.

(a) $\displaystyle\int xe^{-2x}\,dx$

Use integration by parts:

$$u = x, \quad dv = e^{-2x}dx$$

$$du = dx, \quad v = -\frac{1}{2}e^{-2x}$$

Then

$$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

(b) $\displaystyle\int \ln x\,dx$

Use integration by parts:

$$u = \ln x, \quad dv = dx$$

$$du = \frac{1}{x}dx, \quad v = x$$

Then

$$\int \ln x dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x + C$$