MATH 142

MIDTERM EXAM I SOLUTIONS

February 20, 2003

1. (15 pts) Let $f(x) = \dfrac{1+x^2}{1-x^2}$. Then $f'(x) = \dfrac{4x}{(1-x^2)^2}$ and $f''(x) = \dfrac{8x^2+4}{(1-x^2)^3}$.

(a) Find the domain of f.

$$x \neq \pm 1, \text{ i.e. } (-\infty,\ -1) \cup (-1,\ 1) \cup (1,\ \infty)$$

(b) Find all horizontal and vertical asymptotes of f.

There are vertical asymptotes are $x = -1$ and $x = 1$, since

$$\lim_{x \to -1^+} \frac{1+x^2}{1-x^2} = \frac{2}{\text{tiny pos. \#}} = \infty$$

and

$$\lim_{x \to 1^-} \frac{1+x^2}{1-x^2} = \frac{2}{\text{tiny pos. \#}} = \infty.$$

There is a horizontal asymptote at $y = -1$, since

$$\lim_{x \to \infty} \frac{1+x^2}{1-x^2} = -1$$

and

$$\lim_{x \to -\infty} \frac{1+x^2}{1-x^2} = -1.$$

...Continue with $f(x) = \dfrac{1+x^2}{1-x^2}$, $f'(x) = \dfrac{4x}{(1-x^2)^2}$ and $f''(x) = \dfrac{8x^2+4}{(1-x^2)^3}$.

(c) Find the intervals on which f is increasing and the intervals on which f is decreasing.

The denominator of $f'(x)$ is always positive. Thus $f'(x) > 0$ when $x > 0$ and $f'(x) < 0$ when $x < 0$. Therefore $f(x)$ is increasing if $x > 0$, and $f(x)$ is decreasing if $x < 0$.

(d) Find all local extrema for f.

$f'(x) = 0$ only when $x = 0$. So 0 is a critical number. $f'(x)$ is undefined at ± 1, but these don't count as critical numbers since they're not in the domain.

Note that $f''(0) = \frac{4}{1} > 0$, so by the second derivative test, $x = 0$ is a local min.

(e) Find the intervals on which f is concave up and those on which it is concave down.

We need to look at the sign of $f''(x)$. The numerator of $f''(x)$ is always positive, so $f''(x)$ is never 0. However, $f''(x)$ is undefined at ± 1. Thus there are three intervals where we have to check the sign: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. We'll use -2, 0 and 2 as test numbers:

$$f''(-2) = \frac{36}{(-3)^3} < 0, \quad f''(0) = 4 > 0, \quad f''(2) = \frac{36}{(-3)^3} < 0.$$

f is concave up where $f''(x) > 0$, and concave down where $f''(x) < 0$. Thus f is concave up on $(-1,\ 1)$, and concave down on $(-\infty,\ -1) \cup (1,\ \infty)$.

2. (10 pts) Using the information from the previous problem, sketch the graph of

$$y = \frac{1 + x^2}{1 - x^2}$$

on the axes below. Label the coords of all **max/mins** and **intercepts**, and include the equations for all **asymptotes**. (On the axes below, 1 mark = 1 unit.)

3. (10 pts) The width of a certain rectangle is four times the reciprocal of its length. What is the smallest possible value for the perimeter?
Your answer should be the perimeter, not the width!

Let x be the length and w the width. Then $w = \frac{4}{x}$.
The perimeter is $P = 2x + 2w$. Substituting, we have

$$P(x) = 2x + \frac{8}{x}$$

and

$$P'(x) = 2 - \frac{8}{x^2}.$$

so $P'(x) = 0$ if $x = 2$.
Thus $x = 2$ is a critical number. We need to make sure this gives the minimum value. We always have $x \geq 0$, so there are two intervals to check: $(0, 2)$ and $(2, \infty)$. $P'(1) = -6 < 0$ and $P'(3) = 2 - \frac{8}{9} > 0$. Hence $x = 2$ is a minimum by the first derivative test. This also shows that $x = 2$ is the absolute minimum (since $P(x)$ is larger at all values to the left and right of 2).
Thus the smallest possible perimeter is $P(2) = 8$.

4. (5 pts) Suppose we are trying to solve $x^3 - x + 3 = 0$ using Newton's method. If our first guess is $x = 1$, what will our second guess be?

$f(x) = x^3 - x + 3$ and $f'(x) = 3x^2 - 1$.
We start with $x_1 = 1$. Then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{3}{3-1} = -\frac{1}{2}.$$

5. (12 pts) A ball is thrown straight up from the top of an 80 ft building with an initial velocity of 64 ft/sec. (The acceleration due to gravity is -32 ft/sec^2.)

(a) Find the formula for the velocity $v(t)$ at time t.

$v'(t) = a = -32$ and $v(0) = 64$. Hence $v(t) = -32t + 64$.

(b) Find the formula for the height $h(t)$ at time t.

$h'(t) = v(t)$ and $h(0) = 80$, so $h(t) = -16t^2 + 64t + 80$.

(c) When does the ball reach its highest point?

$v(t) = 0$ when the ball reaches its highest point. Setting $v(t) = -32t + 64 = 0$, we find that $t = 2$.

(d) When does the ball land on the ground?

$h(t) = 0$ when the ball hits the ground. Thus $-16t^2 + 64t + 80 = 0$, and we find that $t = 5$ or $t = -1$. So the answer is $t = 5$.

6. (12 pts) Find the most general antiderivatives for the following:

(a) $f(x) = \sqrt{x} + 5x^2 - \dfrac{7}{x}$

$$F(x) = \frac{2}{3}x^{3/2} + \frac{5}{3}x^3 - 7\ln|x| + C$$

(b) $f(x) = \dfrac{5}{1+x^2}$

$$F(x) = 5tan^{-1}(x) + C$$

(c) $f(x) = \sin(2x) + e^{-x}$

$$F(x) = -\frac{1}{2}cos(2x) - e^{-x} + C$$

7. (12 pts) Approximate the value of the integral

$$\int_1^7 (x^2 + 2)dx$$

using the Riemann sum with $n = 3$ rectangles, using right endpoints. You don't have to simplify your answer.

$\Delta x = \frac{7-1}{3} = 2$. Then

$$R_3 = f(3)\Delta x + f(5)\Delta x + f(7)\Delta x = 11 \cdot 2 + 27 \cdot 2 + 51 \cdot 2.$$

8. (12 pts) Evaluate $\int_{-1}^{2} |x| dx$ by drawing a picture and computing the appropriate area.

We add the areas of the two triangles:

$$\int_{-1}^{2} |x| = \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 2 = 5/2$$

9. (12 pts) Find a point on the parabola $y = x^2 - 1$ whose distance from the point $(0, 3)$ is as small as possible.

Let $(x,\ y)$ be any point on the curve. The distance from (x, y) to $(0, 3)$ is

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-3)^2} \\ &= \sqrt{x^2 + (x^2 - 1 - 3)^2} \\ &= \sqrt{x^2 + (x^2 - 4)^2}. \end{aligned}$$

Instead of minimizing d, we know it suffices to minimize d^2. Set

$$f(x) = d^2 = (x-0)^2 + (x^2 - 4)^2.$$

Then

$$f'(x) = 2x + 4x(x^2 - 4) = 2x(2x^2 - 7).$$

We see that $f'(x) = 0$ when $x = 0$ and $x = \pm\sqrt{7/2}$.
$f''(x) = 12x^2 - 14$ and $f''(0) = -14 < 0$, $f(\pm\sqrt{7/2}) = 28 > 0$.
Hence by the second derivative test, 0 is a local max, while $\pm\sqrt{7/2}$ are local mins. (Alternatively, calculating $f(0)$ and $f(\pm\sqrt{7/2})$ one sees that the smallest value of $f(x)$ occurs when $x = \pm\sqrt{7/2}$.) Thus the two acceptable answers are $(\sqrt{7/2},\ 5/2)$ and $(-\sqrt{7/2}, 5/2)$.