Math 142 Spring 2003
Final Exam Solutions

Part I

1. (12 pts) An ant crawls along a straight line with velocity

$$v(t) = 4 - 2t.$$

(a) Find the displacement of the ant between $t = 0$ and $t = 10$.

$$\int_0^{10} 4 - 2t \, dt$$

$$= 4t - t^2 \Big|_0^{10} = (40 - 100) - 0 = -60$$

ANSWER: -60

(b) Find the total distance traveled by the ant between $t = 0$ and $t = 10$.

$$v(t) = 4 - 2t = 0$$

$$t = 2$$

$$\left| \int_0^2 4 - 2t \, dt \right| + \left| \int_2^{10} 4 - 2t \, dt \right|$$

$$\left| 4t - t^2 \Big]_0^2 \right| + \left| 4t - t^2 \Big]_2^{10} \right|$$

$$\left| (8 - 4 - 0) \right| + \left| (40 - 100) - (8 - 4) \right|$$

$$|4| + |-64| = 68$$

ANSWER: 68

2. (25 pts) Let $f(x) = x + \dfrac{1}{x}$.

(a) Find the critical numbers of f.

$$f'(x) = 1 - \frac{1}{x^2} = 0$$

$$1 = \frac{1}{x^2}$$

+4

$$x^2 = 1$$

$$x = \sqrt{1}$$

$$x = \pm 1$$

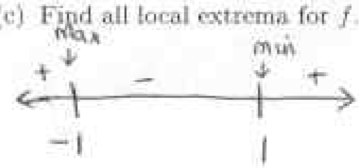
(b) Find the intervals where f is increasing and decreasing.

+4

INC $(-\infty, -1)$ $(1, +\infty)$

DEC $(-1, 1)$

(c) Find all local extrema for f.



+4

$x = -1 \rightarrow$ local max $(-1, 2)$ & $f(-1) = -1 + \frac{1}{-1} = -1 + -1 = -2$

$x = 1$ is local min $(1, 2)$ $f(1) = 1 + \frac{1}{1} = 1 + 1 = 2$

...Continuing with $f(x) = x + \frac{1}{x}$.

(d) Find the intervals where f is concave up/concave down.

$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$

$-$ | $+$

0

CU $(0, \infty)$

CD $(-\infty, 0)$

+4

(e) Show that f has a vertical asymptote at $x = 0$.

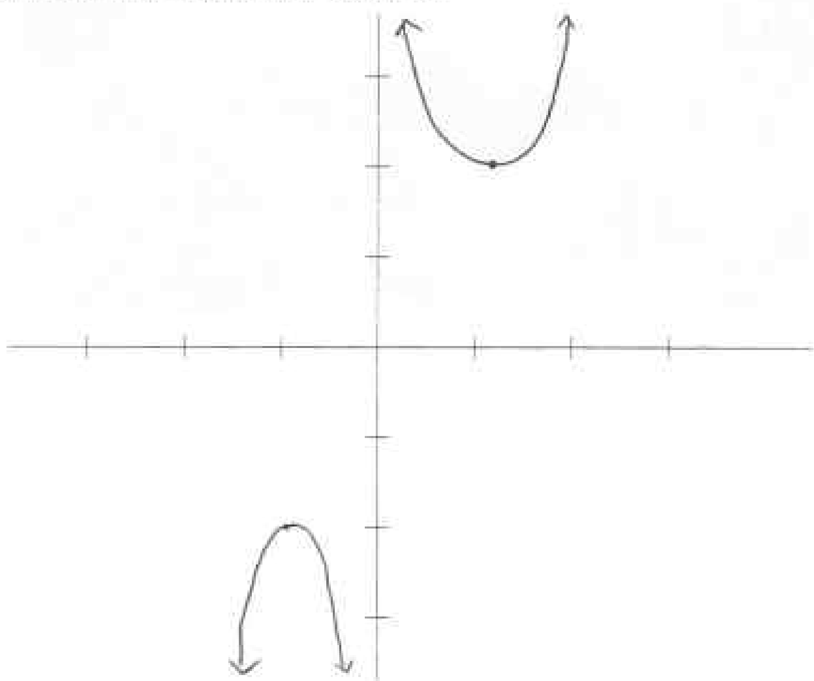
$$\lim_{x \to 0^+} x + \frac{1}{x} = +\infty$$

+4

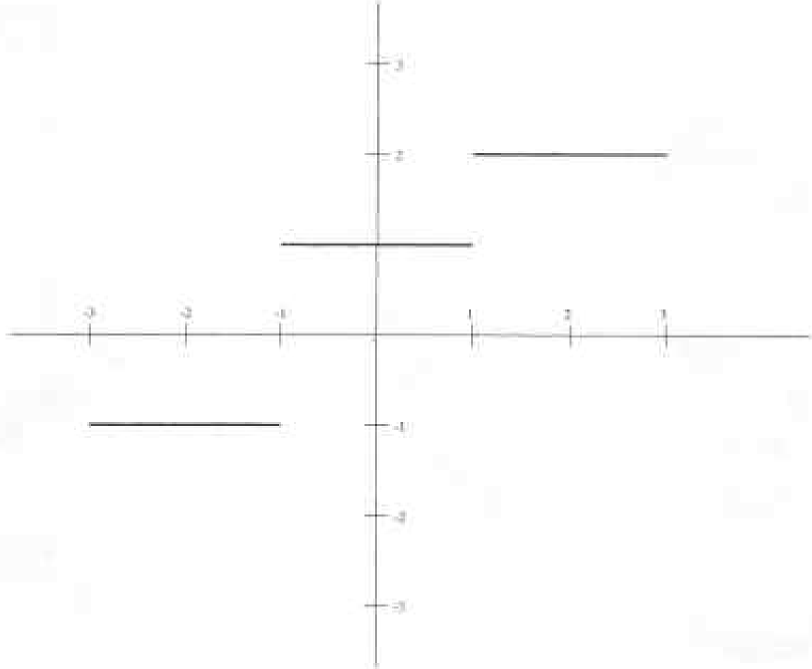
$$\lim_{x \to 0^-} x + \frac{1}{x} = -\infty$$

(f) Graph the curve $y = f(x)$ on the given axes

+5



3. (12 pts) Below is the graph of a (discontinuous) function $y = f(x)$.



Do the following (no need to show any work):

(a) Find $\int_0^3 f(x)dx$. $= \int_0^1 1\, dx + \int_1^3 2\, dx = 1 + 4 = \boxed{5}$

(b) Find $\int_0^3 2f(x)dx$. $= 2\int_0^3 f(x)\, dx = 2 \cdot 5 = \boxed{10}$

(c) Find $\int_{-3}^0 f(x)dx$. $= \int_{-3}^{-1} -1\, dx + \int_{-1}^0 1\, dx = -2 + 1 = \boxed{-1}$

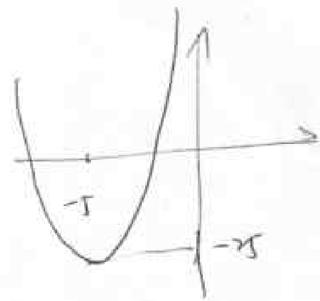
(d) Find $\int_0^{-3} f(x)dx$. $= -\int_{-3}^0 f(x)\, dx = -(-1) = \boxed{1}$

4. (12 pts) The difference of two numbers is 10. What is the smallest possible value for their product? (Show that your answer is the minimum).

Solution:

$$x - y = 10 \qquad x = y + 10$$

$$f(y) = xy = (y+10)\cdot y = y^2 + 10y$$
$$= (y+5)^2 - 25$$



$$f'(y) = 2y + 10 = 0$$

$$\Rightarrow \quad y = -5$$

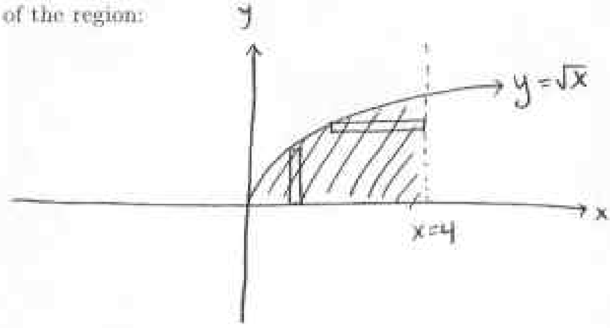
$$\Rightarrow \quad x = 5$$

$$\Rightarrow \quad x \cdot y = -25$$

$$f''(y) = 2 > 0$$

5. (12 pts) Consider the region under the graph of $y = \sqrt{x}$ between $x = 0$ and $x = 4$. A solid is formed by rotating this region around the y-axis.

Draw a picture of the region:



(a) Using dx to compute the volume will result in: **Washers** **Shells** (circle one).

(b) Using dy to compute the volume will result in: **Washers** **Shells** (circle one).

(c) Compute the volume of the solid, using whichever method you prefer.

Shells

$V = 2\pi r h\, dx$

$r = x$

$h = \sqrt{x}$

$\text{Total Volume} = \int_0^4 2\pi x\sqrt{x}\, dx$

$= 2\pi \int_0^4 x^{3/2}\, dx$

$= 2\pi \left[\frac{2}{5}x^{5/2}\right]_0^4$

$= 2\pi \left[\frac{2}{5}(4)^{5/2}\right]$

$= \frac{128\pi}{5}$

Washer

$V = \pi(R^2 - r^2)\, dy$

$R = 4 \qquad r = x = y^2$

$\text{Total Volume} = \pi \int_0^2 4^2 - (y^2)^2\, dy$

$= \pi \int_0^2 16 - y^4\, dy$

$= \pi \left[16y - \frac{1}{5}y^5\right]_0^2$

$= \pi \left[32 - \frac{32}{5}\right]$

$= \frac{128\pi}{5}$

6. (12 pts) A carpet which is 8 meters long is rolled up. When x meters have been unrolled, the force required to unroll it further is

$$F(x) = e^x(8 - x) \text{ Newtons.}$$

How much work does it take to unroll the entire carpet?

$$\begin{aligned} W &= \int_a^b F(x)dx \\ &= \int_0^8 e^x(8 - x)\, dx. \end{aligned}$$

Let

$$\begin{aligned} u = (8 - x); &\quad dv = e^x dx \\ du = -dx; &\quad v = e^x. \end{aligned}$$

Then

$$\begin{aligned} W &= e^x(8 - x)|_0^8 - (-\int_0^8 e^x dx) \\ &= e^x(8 - x) + e^x|_0^8 \\ &= e^8 - 9\ N. \end{aligned}$$

7. (15 pts) Compute the following integrals:

(a) $\displaystyle\int x \sin(x^2)\, dx$

Let $u = x^2$, $du = 2x\, dx$. We get

$$\frac{1}{2}\int \sin(u)\, du = -\frac{1}{2}\cos(u) + C$$
$$= -\frac{1}{2}\cos(x^2) + C.$$

(b) $\displaystyle\int_0^3 \frac{x^2}{\sqrt{x^3+1}} dx$

Let $u = (x^3+1)$, $du = 3x^2\, dx$. When $x = 0$, $u = 1$; when $x = 3$, $u = 28$. Thus we get

$$\frac{1}{3}\int_1^{28} \frac{1}{\sqrt{u}} du = \frac{1}{3}\int_1^{28} u^{-1/2} du$$
$$= \frac{1}{3}\frac{u^{1/2}}{1/2}\Big|_1^{28}$$
$$= \frac{2}{3}\sqrt{28} - \frac{2}{3}$$

(c) $\displaystyle\int x^2 \ln x\, dx$

Let

$$u = \ln x \quad dv = x^2 dx$$
$$du = \frac{1}{x} dx \quad v = x^3/3.$$

We get

$$\frac{x^3}{3}\ln x - \int \frac{x^3}{3x} dx$$
$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$$
$$= \frac{x^3}{3}\ln x - \frac{1}{3}x^3/3 + C$$
$$= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

Part II

1. (10 pts) Compute the following integrals.

(a) $\int \cos^3(x)dx \;=\; \int \cos x \cdot \cos^2 x \, dx = \int \cos x \, (1 - \sin^2 x) dx$

$u = \sin x$

$du = \cos x \, dx$

$$\int \cos^3 x \, dx = \int (1 - u^2)\, du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C$$

(b) $\int \tan^2(x)\sec^4(x)dx \;=\; \int \tan^2 x \, (1 + \tan^2 x) \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$$\int \tan^2(x)\sec^4(x)dx = \int u^2(1+u^2)du = \int (u^2 + u^4)du =$$

$$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

2. (12 pts) An aphid crawled in the plane from $(0,0)$ to $(4, \frac{16}{3})$ along the curve $y = \frac{2}{3}x^{3/2}$. How far did he crawl?

$$\frac{dy}{dx} = x^{1/2}$$

$$\int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^4 \sqrt{1+x}\, dx$$

$u = 1+x \quad x=0 \quad u=1$

$du = dx \quad x=4 \quad u=5$

$$= \int_1^5 u^{1/2}\, du$$

$$= \frac{2}{3}u^{3/2}\Big|_1^5 = \frac{2}{3}\left(5^{3/2}-1\right)$$

3. (16 pts) Compute the following integrals.

(a) (10 points) $\int \frac{x+3}{x^2+9}dx$

$$= \int \frac{x}{x^2+9}dx + \int \frac{3}{x^2+9}dx$$

$$= \int \frac{x}{x^2+9}dx + 3(1/3)\arctan(x/3) + C$$

For the first integral, let $u = x^2+9$, $du = 2x\,dx$.

$$= \frac{1}{2}\int \frac{1}{u}du + \arctan(x/3) + C$$

$$= \frac{1}{2}\ln(x^2+9) + \arctan(x/3) + C.$$

(b) (6 points) $\int \frac{1}{e^x+1}dx$

Let $u = e^x$, $du = e^x dx$. Then $dx = \frac{du}{e^x} = \frac{du}{u}$. Thus we have

$$\int \frac{1}{(u+1)u}du.$$

This requires partial fractions:

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu.$$

Setting $u = 0$, we find $A = 1$. Setting $u = -1$, we find $B = -1$. Thus we have

$$\int \frac{1}{u(u+1)}du = \int \frac{1}{u} - \frac{1}{u+1}du$$

$$= \ln|u| - \ln|u+1| + C$$

$$= \ln|e^x| - \ln|e^x+1| + C$$

$$= x - \ln|e^x+1| + C.$$

4. (20 pts) Compute the following integrals.

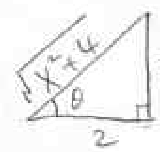
(a) (10 points) $\int \frac{x^3}{\sqrt{x^2+4}}dx$

Let $X = 2\tan\theta$, $dX = 2\sec^2\theta\, d\theta$

$\sqrt{X^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4\sec^2\theta} = 2\sec\theta$

$$\int \frac{X^3}{\sqrt{X^2+4}}dX = \int \frac{8\tan^3\theta \cdot 2\sec^2\theta\, d\theta}{2\sec\theta} = 8\int \tan^3\theta \sec\theta\, d\theta$$

Let $u = \sec\theta$, $du = \tan\theta\sec\theta\, d\theta$.

$\left(u = \sec\left(\arctan\left(\frac{X}{2}\right)\right) = \frac{\sqrt{X^2+4}}{2} \right.$  $)$

So $8\int \tan^3\theta \sec\theta\, d\theta$

$= 8\int \tan^2\theta \cdot (\tan\theta \cdot \sec\theta)\, d\theta$

$= 8\int (\sec^2\theta - 1)\cdot(\tan\theta \cdot \sec\theta)\, d\theta$

$= 8\int (u^2-1)\, du = \frac{8}{3}u^3 - 8u + C$

$= \frac{1}{3}\left(\sqrt{X^2+4}\right)^3 - 4\sqrt{X^2+4} + C$

(or: $\frac{1}{3}\sqrt{X^2+4}\left(X^2-8\right) + C$)

(b) (10 points) $\int \frac{2x+3}{x^2(x-1)}dx$

$$\frac{2X+3}{X^2(X-1)} = \frac{A}{X} + \frac{B}{X^2} + \frac{C}{X-1} = \frac{AX(X-1)+B(X-1)+CX^2}{X^2(X-1)}$$

$\therefore \; A(X-1)X + B(X-1) + CX^2 = 2X+3$.

Plug in $X=0 \Rightarrow -B = 3, \quad B=-3$;

" $\quad X=1 \Rightarrow C=5$;

" $\quad X=2 \Rightarrow 2A - 3 + 5 \cdot 4 = 7, \quad A=-5$.

So $\int \frac{2X+3}{X^2(X-1)}dX = \int \frac{-5}{X}dx + \int \frac{-3}{X^2}dX + \int \frac{5}{X-1}dX$

$$= -5\ln|x| + \frac{3}{X} + 5\ln|X-1| + C$$

$$= 5\ln\left|\frac{X-1}{X}\right| + \frac{3}{X} + C$$

5. (10 pts) An asteroid is moving in a straight line. Its velocity is measured at 2 second intervals. These measurements are tabulated below:

Time	km/sec
0	5.1
2	4.1
4	3.5
6	2.0
8	1.2

Use Simpson's rule to estimate the distance traveled by the asteroid between $t = 0$ and $t = 8$. You don't need to simplify.

$\Delta x = 2$

$$S = \frac{2}{3}\left[5.1 + 4(4.1) + 2(3.5) + 4(2.0) + 1.2\right]$$

6. (12 pts) Evaluate the following integrals. Show your work.

(a) $\displaystyle\int_2^{\infty} \frac{1}{x^2} dx$

$$\begin{aligned} &= \lim_{t\to\infty} \int_2^t x^{-2} dx \\ &= \lim_{t\to\infty} \left(-x^{-1}\right|_2^t \\ &= \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{2}\right) \\ &= \frac{1}{2}. \end{aligned}$$

(b) $\displaystyle\int_0^{2} \frac{1}{x^2} dx$

$$\begin{aligned} &= \lim_{t\to 0+} \int_t^2 x^{-2} dx \\ &= \lim_{t\to 0+} \left(-\frac{1}{x}\right|_t^2 \\ &= \lim_{t\to 0+} \left(-\frac{1}{2} + \frac{1}{t}\right) \\ &= \left(-\frac{1}{2} + \infty\right) \end{aligned}$$

Hence the integral is divergent.

7. (20 pts) A flat plastic plate has the shape of the region under the graph of $y = \sin x$ between $x = 0$ and $x = \pi$.

(a) Find the area of the plate.

$$A = \int_0^\pi \sin x \, dx$$

4 pts

$$= -\cos(\pi) + \cos(0) = \boxed{2}$$

(b) Find the x-coordinate of the center of mass of the plate.

$$\bar{x} = \frac{1}{A} \int_0^\pi x \sin x \, dx$$

8 pts

$$= \frac{1}{2} \int_0^\pi x \sin x \, dx$$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$$= \frac{1}{2}\left[-x\cos x \Big|_0^\pi - \int_0^\pi -\cos x \, dx \right]$$

$$= \frac{1}{2}\left[-\pi(-1) + (\sin(\pi) - \sin(0)) \right]$$

$$= \boxed{\frac{\pi}{2}}$$

(c) Find the y-coordinate of the center of mass of the plate.

8 pts

$$\bar{y} = \frac{1}{A} \int_0^\pi \frac{1}{2} \sin^2 x \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{2} \int_0^\pi \frac{1 - \cos(2x)}{2} \, dx$$

$$= \frac{1}{8} \int_0^\pi 1 - \cos(2x) \, dx$$

$$= \frac{1}{8}\left[x - \frac{1}{2}\sin(2x) \right|_0^\pi$$

$$= \boxed{\frac{\pi}{8}}$$