MATH 142

Midterm 2 ANSWERS

January 10, 2003

1. (30 pts) Solve the following integrals.

(a) (10 pts)

$$\int 3\sin(2x)dx$$

$$= -\frac{3}{2}\cos(2x) + C$$

(b) (10)

$$\int \frac{2}{\sqrt{x}}\,dx$$

$$= \int 2x^{-1/2}dx + C = 4x^{1/2} + C = 4\sqrt{x} + C$$

(c) (10)

$$\int_{-4}^{-2} \frac{2+x}{5x}\,dx$$

$$\begin{aligned}
&= \frac{2}{5}\int_{-4}^{-2}\frac{1}{x}\,dx + \frac{1}{5}\int_{-4}^{-2}\,dx \\
&= \frac{2}{5}\ln|x|\Big|_{-4}^{-2} + \frac{1}{5}x\Big|_{-4}^{-2} \\
&= \frac{2}{5}(\ln 2 - \ln 4) + \frac{1}{5}\Big(-2-(-4)\Big) \\
&= -\frac{2}{5}\ln 2 + \frac{2}{5}
\end{aligned}$$

2. (26 pts)

(a) (13 pts) Find

$$\frac{d}{dx}\int_0^{\sqrt{x}}(1+t^4)dt$$

Let

$$F(y) = \int_0^y (1+t^4)dt$$

Then $F'(y) = 1 + y^4$ by the fundamental theorem of calculus. Thus, the answer to our problem is

$$\frac{d}{dx}F(\sqrt{x}).$$

But by the chain rule,

$$\begin{aligned}
\frac{d}{dx}F(\sqrt{x}) &= F'(\sqrt{x})\left[\sqrt{x}\right]' \\
&= F'(\sqrt{x})\left[x^{1/2}\right]' \\
&= \left(1 + \sqrt{x}^{\,4}\right)\frac{1}{2}x^{-1/2} \\
&= \frac{1}{2}\left(1 + x^2\right)x^{-1/2} \\
&= \frac{(1 + x^2)}{2x^{1/2}}
\end{aligned}$$

(b) (13 pts) Find

$$\frac{d}{dx}\int_{x^2}^{0} \sin^4(t)\,dt$$

Let $F(y)$ be an antiderivative of $\sin^4(y)$, so that $F'(y) = \sin^4(y)$. Then,

$$\int_{x^2}^{0} \sin^4(t)\,dt = F(0) - F(x^2)$$

and by the chain rule,

$$\begin{aligned}
\frac{d}{dx}\int_{x^2}^{0} \sin^4(t)\,dt &= \frac{d}{dx}\left(F(0) - F(x^2)\right) \\
&= -\frac{d}{dx}F(x^2) \\
&= -F'(x^2)2x \\
&= -\sin^4(x^2)2x
\end{aligned}$$

3. (15 pts) Suppose that you keep track of the rainfall, in inches per hour, for Rochester. Time is measured in hours. It is now time 0. At time t, it is raining at $(t^2 + t)/10,000$ inches per hour. Find the amount of rainfall over a 3-day period, starting now.
Hint: How many hours are in 3 days?

3 days is 72 hours, so the total rainfall would be

$$\int_0^{72} \frac{t^2 + t}{10,000}\,dt = \frac{1}{10,000}\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Bigg|_0^{72} = \frac{1}{10,000}\left(\frac{72^3}{3} + \frac{72^2}{2}\right) = 12.7 \text{ inches.}$$

4. (39 pts) Solve the following integrals.

(a) (13 pts)

$$\int \frac{\sin(\ln(x))}{x}\, dx$$

Let

$$\begin{aligned} u &= \ln(x) \\ du &= \frac{1}{x}\, dx \end{aligned}$$

Then,

$$\int \frac{\sin(\ln(x))}{x}\, dx = \int \sin(u)\, du = -\cos(u) + C = -\cos(\ln(x)) + C$$

(b) (13 pts)

$$\int_0^{\pi/4} \cos(2x) e^{\sin(2x)}\, dx$$

Let

$$\begin{aligned} u &= \sin(2x) \\ du &= 2\cos(2x) dx \end{aligned}$$

Then, since $\sin(0) = 0$ and $\sin(2 \cdot \pi/4) = \sin(\pi/2) = 1$,

$$\int_0^{\pi/4} \cos(2x) e^{\sin(2x)}\, dx = \frac{1}{2}\int_0^1 e^u\, du = \frac{1}{2} e^u \Big|_0^1 = \frac{1}{2}\left(e^1 - e^0\right) = \frac{e-1}{2}$$

(c) (13 pts)

$$\int_0^1 \frac{e^x}{e^x + 1}\, dx$$

Let

$$\begin{aligned} u &= e^x + 1 \\ du &= e^x dx \end{aligned}$$

Then, since $e^0 + 1 = 2$ and $e^1 + 1 = e + 1$,

$$\int_0^1 \frac{e^x}{e^x+1}\, dx = \int_2^{e+1} \frac{1}{u}\, du = \ln|u| \Big|_2^{e+1} = \ln(e+1) - \ln(2) = \ln\left(\frac{e+1}{2}\right)$$

5. (15 pts) Find the area between the curves

$$\begin{aligned} y &= 3x + 3 \\ y &= 3 - x^2 \end{aligned}$$

between $x = 0$ and $x = 1$.

Since $y = 3 - x^2$ is below $y = 3x + 3$ on the interval $(0, 1)$, the area is

$$\begin{aligned} A &= \int_0^1 \left(3x + 3 - 3 + x^2\right) dx \\ &= \int_0^1 \left(3x + x^2\right) dx \\ &= \left(\frac{3x^2}{2} + \frac{x^3}{3}\right)\Big|_0^1 \\ &= \frac{3}{2} + \frac{1}{3} \\ &= \frac{11}{6} \end{aligned}$$

6. (15 pts) Find the area between the curves

$$\begin{aligned} y &= x^2 - 1 \\ y &= x + 1 \end{aligned}$$

Hint: Find the points at which the curves intersect.

Setting the right hand sides of the equations equal, we get

$$x^2 - 1 = x + 1$$

or

$$0 = x^2 - x - 2 = (x - 2)(x + 1)$$

So, the curves intersect at $x = -1, 2$. Checking at $x = 0$, we see that the curve $y = x + 1$ is on top, so the area would be

$$\begin{aligned} A &= \int_{-1}^2 \left(-x^2 + x + 2\right) dx \\ &= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Big|_{-1}^2 \\ &= \left(-\frac{8}{3} + \frac{4}{2} + 4\right) - \left(-\frac{-1}{3} + \frac{1}{2} - 2\right) \\ &= -\frac{9}{3} + 2 + 4 - \frac{1}{2} + 2 \\ &= 5 - \frac{1}{2} \\ &= \frac{9}{2} \end{aligned}$$

7. (15 pts) Find the volume of the solid obtained by rotating the region bounded by the given curves, about the x-axis.

$$\begin{aligned} y &= x^2 \\ y &= 2x \end{aligned}$$

We find the intersection points by setting the right hand sides of the equations equal, to get $x^2 = 2x$. The points are $x = 0, 2$. We use vertical slices, and the slices are washers. The area of the cross-section is

$$A(x) = \pi(2x)^2 - \pi(x^2)^2.$$

Thus, the volume is

$$\begin{aligned} V &= \int_0^2 \left(\pi(2x)^2 - \pi(x^2)^2\right) dx \\ &= \pi \int_0^2 \left(4x^2 - x^4\right) dx \\ &= \pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 \\ &= \pi \left(\frac{4}{3} \cdot 8 - \frac{32}{5}\right) \\ &= \pi \left(\frac{32}{3} - \frac{32}{5}\right) \\ &= 32\pi \left(\frac{1}{3} - \frac{1}{5}\right) \\ &= 32\pi \frac{2}{5} \\ &= \frac{64}{15}\pi \end{aligned}$$

8. (15 pts) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.
****WARNING** Unless you use the shell method, you will not get full credit.**

$$\begin{aligned} y &= e^{x^2} \\ y &= 0 \\ x &= 0 \\ x &= 4 \end{aligned}$$

The volume of a cylindrical shell is $2\pi hr\Delta r$, where h is the height and r is the radius. In this case, $r = x$ and $h = e^{x^2}$.

$$V = \int_0^4 2\pi x e^{x^2}\, dx$$

Use the substitution $u = x^2$, $du = 2xdx$. The limits of $x = 0$, $x = 4$ become $u = 0$, $u = 16$.

$$\begin{aligned} V &= \pi \int_0^{16} e^u\, du \\ &= \pi e^u \Big|_0^{16} \\ &= \pi \left(e^{16} - 1\right). \end{aligned}$$

9. (15 pts) A spring at rest has length of 1 meter. Assuming that the spring constant k equals 10 Newtons per meter squared. Calculate the work required to stretch the spring so as to increase its length to 3 meters.

Hooke's law says that the force you need to use on the spring is

$$F = kx = 10x.$$

where x is the distance beyond equilibrium. Since we are starting from an equilibrium position of 1, we must integrate from $x = 0$ to $x = 2$. The work is

$$\begin{aligned} W &= \int_0^2 10x\, dx \\ &= 10 \int_0^2 x\, dx \\ &= 10 \cdot \frac{x^2}{2}\Big|_0^2 \\ &= 20 \text{ Joules.} \end{aligned}$$

10. (15 pts) Find the average value of $f(x) = x\sqrt{1+x^2}$ over $[0, 4]$.

The average value would be

$$\frac{1}{4-0} \int_0^4 x\sqrt{1+x^2}\, dx$$

Use the substitution $u = 1 + x^2$, $du = 2xdx$. The limits of integration, which were $x = 0$ and $x = 4$ now become $u = 1$ and $u = 17$. Then, the average value of f is

$$\begin{aligned} \frac{1}{4} \int_1^{17} \sqrt{u}\, \frac{du}{2} &= \frac{1}{8} \int_1^{17} u^{1/2}\, du \\ &= \frac{1}{8} \cdot \frac{2}{3} u^{3/2}\Big|_1^{17} \\ &= \frac{1}{12}\left(17^{3/2} - 1\right). \end{aligned}$$