MATH 142

Midterm 1 ANSWERS

January 10, 2003

1. (10 pts) Suppose that we are trying to solve

$$x^4 + x - 1 = 0$$

using Newton's method. Assume that our first guess was $x = 1$. What would the second guess be?

$$\begin{array}{ll} f(x) = x^4 + x - 1 & f(1) = 1 \\ f'(x) = 4x^3 + 1 & f'(1) = 5 \end{array}$$

Therefore, setting $x_0 = 1$, our second guess would be x_1, where

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{1}{5} \\ &= \frac{4}{5}. \end{aligned}$$

2. (14 pts) A cylindrical can must have a volume of 10 cubic inches. The top and bottom cost 1 cent per square inch, while the side costs 2 cents per square inch. Find the dimensions of the can which minimizes the cost.

Let r be the radius of the can, and let h be the height. Since the volume must be 10, we have

$$10 = \pi r^2 h \qquad \text{and so} \qquad h = \frac{10}{\pi} \cdot \frac{1}{r^2}$$

The area of the top and bottom is

$$2\pi r^2$$

and the cost is

$$2\pi r^2 \cdot 1$$

The area of the side is

$$2\pi r h$$

and its cost is

$$2\pi r h \cdot 2$$

We want to minimize the cost, which is

$$\begin{aligned} f(r) &= 2\pi r h \cdot 2 + 2\pi r^2 \cdot 1 \\ &= 4\pi r \frac{10}{\pi} \frac{1}{r^2} + 2\pi r^2 \\ &= \frac{40}{r} + 2\pi r^2 \end{aligned}$$

To find the critical points, we set $f'(r) = 0$, and get

$$0 = f'(r) = -\frac{40}{r^2} + 4\pi r$$

Therefore,

$$\frac{40}{r^2} = 4\pi r$$

and

$$r^3 = \frac{40}{4\pi} = \frac{10}{\pi}$$

so

$$r = \left(\frac{10}{\pi}\right)^{1/3}$$

Common sense shows that $r \in (0, \infty)$, and that the minimum cost cannot occur at the endpoints. Therefore the minimum cost occurs at the value of r computed above. Solving for h, we find that the dimensions of the can which minimizes cost are

$$\begin{aligned} r &= \left(\frac{10}{\pi}\right)^{1/3} \\ h &= \frac{10}{\pi} \cdot \frac{1}{r^2} = \frac{10}{\pi} \cdot \left(\frac{\pi}{10}\right)^{2/3} = \left(\frac{10}{\pi}\right)^{1/3} \end{aligned}$$

3. (14 pts) Find the point on the line $y = x$ which is closest to $(1, 2)$.

We are trying to minimize the distance

$$D(x) = \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x-1)^2 + (x-2)^2}$$

where we have used the equation $y = x$ to substitute for y. We may as well try to minimize the square of the distance, namely

$$f(x) = (x-1)^2 + (x-2)^2$$

We have $x \in (-\infty, \infty)$, and common sense tells us that the minimum does not occur at $\pm\infty$. We find the critical points by taking the derivative of f and setting it equal to 0.

$$0 = f'(x) = 2(x-1) + 2(x-2) = 2(2x-3)$$

Therefore, $x = 3/2$. Since $y = x$, we have $y = 3/2$, and the closest point is

$$(3/2, 3/2)$$

4. (14 pts) Consider the curve defined by

$$y = \frac{x^2 - 4x + 1}{2x^2 + 5x - 3}.$$

(a) (7 pts) Find the vertical asymptotes.

There will be a vertical asymptote where the denominator blows up. We can factor the denominator as follows.

$$2x^2 + 5x - 3 = (2x-1)(x+3)$$

So, the vertical asymptotes are at $x = -3$ and $x = 1/2$.

(b) (7 pts) Find the horizontal asymptotes.

By dropping all but the leading terms, we compute

$$\begin{aligned} \lim_{x\to\infty} \frac{x^2 - 4x + 1}{2x^2 + 5x - 3} &= \lim_{x\to\infty} \frac{x^2}{2x^2} = \frac{1}{2} \\ \lim_{x\to-\infty} \frac{x^2 - 4x + 1}{2x^2 + 5x - 3} &= \lim_{x\to-\infty} \frac{x^2}{2x^2} = \frac{1}{2} \end{aligned}$$

Therefore, the horizontal asymptotes are at $1/2$.

5. (14 pts) Let

$$f(x) = x^4 - 2x^3 - 12x^2 + 3x - 2$$

(a) (7 pts) Find the intervals on which $f(x)$ is concave up and concave down.

First we compute the second derivative:

$$\begin{aligned} f'(x) &= 4x^3 - 6x^2 - 24x + 3 \\ f''(x) &= 12x^2 - 12x - 24 \\ &= 12\left(x^2 - x - 2\right) \\ &= 12(x+1)(x-2) \end{aligned}$$

The zeros of $f''(x)$ are at $x = -1, 2$. These points divide the real line into 3 intervals, $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$. Checking the sign of $f''(x)$ on these 3 intervals, and remembering that $f''(x) > 0$ means that the function is concave up, we find that $f(x)$ is **concave up** on $(-\infty, -1)$ and $(2, \infty)$. It is **concave down** on $(-1, 2)$.

(b) (7 pts) Find the points of inflection.

The points of inflection are where $y = f(x)$ changes from being concave up to concave down, or vice versa. From part (a), the points of inflection are at $x = -1, 2$.

6. (14 pts) Let

$$f(x) = \frac{x+1}{x^2+x+1}$$

(a) (6 pts) Find the intervals of increase and decrease for $y = f(x)$.

Using the quotient rule, we find that

$$\begin{aligned} f'(x) &= \frac{(x^2+x+1)-(x+1)(2x+1)}{(x^2+x+1)^2} \\ &= \frac{x^2+x+1-2x^2-2x-x-1}{(x^2+x+1)^2} \\ &= \frac{-x^2-2x}{(x^2+x+1)^2} \\ &= -\frac{x(x+2)}{(x^2+x+1)^2} \end{aligned}$$

This fraction is 0 exactly when the numerator is 0. Therefore, the critical points are solutions of $x(x+2) = 0$, or $x = -2, 0$. These 2 points cut the real line into 3 intervals, $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$. Checking the sign of $f'(x)$ on these 3 intervals, and remembering that $f'(x) > 0$ when $f(x)$ is increasing, we find that $f(x)$ is **increasing** on $(-2, 0)$. It is **decreasing** on $(-\infty, -2)$ and $(0, \infty)$.

(b) (4 pts) Find the points x at which $f(x)$ has a local maximum.

A local maximum occurs where $f(x)$ switches from being increasing to decreasing. By part (a), this happens at one point, $x = 0$, so this is the **local maximum**.

(c) (4 pts) Find the points x at which $f(x)$ has a local minimum.

A local minimum occurs where $f(x)$ switches from being decreasing to increasing. By part (a), this happens at one point, $x = -2$, so this is the **local minimum**.

7. (18 pts) Find the most general antiderivatives of the following functions.

(a) (6 pts)

$$f(x) = x^{1/2} + x^{-1/2}$$

The antiderivative is:

$$\frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

(b) (6 pts)

$$f(x) = e^x + x^2$$

The antiderivative is:

$$e^x + \frac{1}{3}x^3$$

(c) (6 pts)

$$f(x) = 4\sin(2x) + 5\cos(x)$$

The antiderivative is:

$$-2\cos(2x) + 5\sin(x)$$

8. (10 pts) Assuming that

$$\begin{aligned} f''(x) &= 2x + e^{-x} \\ f(0) &= 1 \\ f'(0) &= 4 \end{aligned}$$

find $f(x)$.

Taking the antiderivative of $f''(x)$, we find that

$$f'(x) = x^2 - e^{-x} + C$$

But since $f'(0) = 4$, we deduce that $4 = -1 + C$, so that $C = 5$, and

$$f'(x) = x^2 - e^{-x} + 5$$

Next, we take the antiderivative of $f'(x)$ to get

$$f(x) = \frac{1}{3}x^3 + e^{-x} + 5x + C$$

Since $f(0) = 1$, we find that $1 = 1 + C$, or $C = 0$. Thus,

$$f(x) = \frac{1}{3}x^3 + e^{-x} + 5x$$

9. (12 pts) A particle moves in a straight line, with an acceleration of $a(t) = 2$ ft/sec^2. Let $s(t)$ be the particle's position at time t and let $v(t)$ be its velocity. If $s(1) = 10$ ft, and $v(1) = 8$ ft/sec^2,

(a) (6 pts) Find $v(t)$.

Taking the antiderivative of $a(t) = 2$, we find that

$$v(t) = 2t + C$$

But since $v(1) = 8$, we have $8 = 2 \cdot 1 + C$, so $C = 6$, and

$$v(t) = 2t + 6\frac{\text{ft}}{\text{sec}}$$

(b) (6 pts) Find $s(t)$.

Taking the antiderivative of $2t + 6$, we find that

$$s(t) = t^2 + 6t + C$$

But since $s(1) = 10$, we have $10 = 1 + 6 + C$, so $C = 3$, and

$$s(t) = t^2 + 6t + 3 \text{ ft}$$

10. (10 pts) Find the Riemann sum corresponding to $n = 4$ for the integral

$$\int_2^6 x^2\,dx$$

using left hand endpoints as sample points.

Since $n = 4$, there are 4 intervals, each of length $\Delta x = 1$. Their left endpoints would be $x = 2, 3, 4, 5$. The corresponding function values would be $x^2 = 4, 9, 16, 25$. The Riemann sum would be

$$\sum_{k=1}^{4} f(x_k)\Delta x = 4 \cdot 1 + 9 \cdot 1 + 16 \cdot 1 + 25 \cdot 1 = 54$$

11. (10 pts) Evaluate the integral by interpreting it in terms of areas.

$$\int_1^3 (1 + 3x)\, dx$$

The region is a trapezoid, with base 2 and heights 4 and 10. Therefore, the integral is given by the area of the trapezoid, which is

$$2 \cdot \frac{4 + 10}{2} = 14$$

12. (10 pts) Let

$$\begin{aligned} \int_2^8 f(x)\, dx &= 5 \\ \int_6^8 f(x)\, dx &= 2 \\ \int_4^6 f(x)\, dx &= 1 \end{aligned}$$

Find

$$\int_2^4 f(x)\, dx$$

By the properties of the integral,

$$\begin{aligned} \int_2^4 f(x)\, dx &= \int_2^8 f(x)\, dx - \int_6^8 f(x)\, dx - \int_4^6 f(x)\, dx \\ &= 5 - 2 - 1 \\ &= 2 \end{aligned}$$