MATH 142

Final ANSWERS

January 10, 2003

Part A

1. (16 pts) Let

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$$

(a) (5 points) Find the intervals on which $f(x)$ is increasing and decreasing.

$$f'(x) = x^2 - 4x + 3 = (x-1)(x-3)$$

Thus, $f(x)$ is increasing in $(-\infty, 1) \cup (3, \infty)$ and $f(x)$ is decreasing in $(1, 3)$.

(b) (5 points) Find the local extrema of $f(x)$.

$x = 1$ is a local maximum and $x = 3$ is a local minimum.

(c) (6 points) Find the intervals on which f is concave up and concave down.

$f''(x) = 2x - 4$. Thus, $f(x)$ is concave up $x > 2$ and concave down $x < 2$.

2. (14 pts)

Let

$$y = \frac{x^2 - 2x + 2}{2x^2 - 5x + 3}.$$

(a) (7 points) Find the vertical asymptotes.

$$y = \frac{x^2 - 2x + 2}{(2x-3)(x-1)}.$$

Thus, vertical asymptotes are $x = 1$ and $x = 1.5$

(b) (7 points) Find the horizontal asymptotes.

$$\lim_{x \to \infty} \frac{x^2 - 2x + 2}{2x^2 - 5x + 3} = \frac{1}{2}$$

3. (20 pts)

A box with a square base and open top must have a volume $32m^3$. Find the dimensions of the box that minimizes the amount of material used. Let x be the width and y be the height. Then $V = x^2y = 32$ and $y = \frac{32}{x^2}$. The surface area of the box will be

$$A(x) = x^2 + 4xy = x^2 + 4x\frac{32}{x^2} = x^2 + \frac{8}{x}.$$

We need to differentiate $A(x)$ and set it equal to 0. Thus

$$A'(x) = 2x - \frac{128}{x^2}$$

We find that $x = 4m$ and $y = 2m$.

4. (16 pts)

Differentiate the following functions.

(a) (8 points)

$$\int_x^0 e^{-2t^2}\,dt$$

$$\frac{d}{dx}\int_x^0 e^{-2t^2}\,dt = -\frac{d}{dx}\int_0^x e^{-2t^2}\,dt = -e^{-2x^2}$$

(b) (8 points)

$$\int_0^{x^3} \sin(t^2)\,dt$$

$$\frac{d}{dx}\int_0^{x^3} \sin(t^2)\,dt = \sin((x^3)^2)\frac{d(x^3)}{dx} = 3x^2\sin(x^6)$$

5. (24 pts)

Evaluate the following integrals.

(a) (8 points)

$$\int (x^2 - e^{2x} + \cos(3x))\,dx$$

$$\begin{aligned}\int (x^2 - e^{2x} + \cos(3x))\,dx &= \int x^2\,dx - \int e^{2x}\,dx + \int \cos(3x)\,dx \\ &= \frac{x^3}{3} - \frac{e^{2x}}{2} + \frac{\sin(3x)}{3} + C\end{aligned}$$

(b) (8 points)

$$\int \frac{dx}{x\ln(2x)}$$

Use the substitution

$$\begin{aligned} u &= \ln(2x) \\ du &= \frac{dx}{x} \end{aligned}$$

Then

$$\int \frac{dx}{x\ln(2x)} = \int \frac{du}{u} = \ln|u| + C = \ln|\,(\ln|2x|)\,| + C$$

(c) (8 points)

$$\int_0^{\ln(\pi/4)} e^x \cos(e^x)\, dx$$

Use the substitution

$$\begin{aligned} u &= e^x \\ du &= e^x\, dx \end{aligned}$$

Then

$$\int_0^{\ln(\pi/4)} e^x \cos(e^x)\, dx = \int_1 \pi/4 \cos(u)\, du = \sin(pi/4) - \sin(1) = \frac{\sqrt{2}}{2} - \sin(1).$$

6. (20 pts)

Find the area between the curves

$$y = x^2, \quad y = x, \quad x = 0, \quad x = 2$$

Note that $x \geq x^2$ in $[0,1]$ and $x^2 \geq x$ in $[1,2]$. Then

$$A = \int_0^1 (x - x^2)\, dx + \int_1^2 (x^2 - x)\, dx = \frac{1}{2} - \frac{1}{3} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = 1.$$

7. (20 pts)

Find the volume of the solid obtained by rotating about the line $y = 2$, the region enclosed by the curves

$$y = \sqrt{x}$$

and

$$y = x.$$

These two functions intersect at points $x = 0$ and $x = 1$. Use washer rule. The radius of $y = x$ at point x is $2 - x$, and the radius of $y = \sqrt{x}$ at point x is $2 - \sqrt{x}$.

$$V = \pi \int_0^1 ((2 - x)^2 - (2 - \sqrt{x})^2)\, dx = \pi \int_0^1 (x^2 - 3x + 4\sqrt{x}) dx = 1.5\pi.$$

8. (20 pts)

A spring has a natural length of $0.1m$. If a $20N$ force is required to keep it stretched to a length $0.3m$, how much work is required to stretch it from $0.1m$ to $0.2m$?

Let x be the distance from the equilibrium position of the sprint. By Hooke's Law, if k is the spring constant, then $20 = 0.2k$. We obtain that $k = 100\frac{N}{m}$. Then, the force at position x is $100x$. Integrating $100x$ from $x = 0.1m$ to $x = 0.2m$, we get

$$W = \int_0^{0.1} 100x\, dx = 50(0.1)^2 = 0.5J.$$

Part B

9. (16 pts) Solve the following integrals.

(a) (8 points)

$$\int x^2 e^{-2x}\, dx$$

Use integration by parts. Let

$$\begin{aligned} u &= x^2 & dv &= e^{-2x}\, dx \\ du &= 2x\, dx & v &= -\frac{1}{2}e^{-2x} \end{aligned}$$

Then, since $\int udv = uv - \int vdu$,

$$\begin{aligned} \int x^2 e^{-2x}\, dx &= -\frac{1}{2}x^2 e^{-2x} - \int \left(-\frac{1}{2}e^{-2x}2x\right) dx \\ &= -\frac{1}{2}x^2 e^{-2x} + \int xe^{-2x}\, dx \end{aligned}$$

We must use integration by parts again, to evaluate the second integral. Let

$$\begin{aligned} u &= x & dv &= e^{-2x}\, dx \\ du &= dx & v &= -\frac{1}{2}e^{-2x} \end{aligned}$$

so that

$$\begin{aligned} \int x^2 e^{-2x}\, dx &= -\frac{1}{2}x^2 e^{-2x} + \int xe^{-2x}\, dx \\ &= -\frac{1}{2}x^2 e^{-2x} + x\left(-\frac{1}{2}e^{-2x}\right) - \int \left(-\frac{1}{2}e^{-2x}\right) dx \\ &= -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C \end{aligned}$$

(b) (8 points)

$$\int \frac{\ln x}{x^3}\,dx$$

We use integration by parts again. Let

$$\begin{aligned} u = \ln x \qquad & dv = x^{-3}\,dx \\ du = \frac{1}{x}\,dx \qquad & v = -\frac{1}{2}x^{-2} \end{aligned}$$

Then

$$\begin{aligned} \int \frac{\ln x}{x^3}\,dx &= -\frac{1}{2}x^{-2}\ln x - \int \left(-\frac{1}{2}x^{-2}\frac{1}{x}\right)\,dx \\ &= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}\,dx \\ &= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C \end{aligned}$$

10. (28 pts)

(a) (9 points) Find

$$\int \sin^2(x)\cos^3(x)\,dx$$

Since the power of $\cos(x)$ is odd, we can use the substitution

$$\begin{aligned} u &= \sin(x) \\ du &= \cos(x)\,dx \end{aligned}$$

So,

$$\begin{aligned} \int \sin^2(x)\cos^3(x)\,dx &= \int \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx \\ &= \int u^2\left(1-u^2\right)\,du \\ &= \int \left(u^2-u^4\right)\,du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C \end{aligned}$$

(b) (10 points) Find

$$\int \sin^2(x)\cos^2(x)\,dx$$

We need to use the formulas

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

to get

$$\int \sin^2(x)\cos^2(x)\,dx = \frac{1}{4}\int \sin^2(2x)\,dx$$

Next, we have to use the formula

$$\sin^2(x) = \frac{1}{2}(1+\cos(2x))$$

to get

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \frac{1}{4}\int \sin^2(2x)\,dx \\
&= \frac{1}{4}\cdot\frac{1}{2}\int (1+\cos(4x))\,dx \\
&= \frac{1}{8}\left(x+\frac{1}{4}\sin(4x)\right)+C \\
&= \frac{1}{8}x+\frac{1}{32}\sin(4x)+C.
\end{aligned}$$

(c) (9 points) Find

$$\int \tan(x)\sec^3(x)\,dx$$

Since the power of $\tan(x)$ is odd, we can make the substitution

$$\begin{aligned}
u &= \sec(x) \\
du &= \tan(x)\sec(x)\,dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \tan(x)\sec^3(x)\,dx &= \int u^2\,du \\
&= \frac{u^3}{3}+C \\
&= \frac{\sec^3(x)}{3}+C.
\end{aligned}$$

11. (18 pts) Solve the following integrals.

(a) (9 points)

$$\int \frac{x^3}{\sqrt{25-x^2}}\,dx$$

We make the substitution

$$\begin{aligned}
x &= 5\sin(u) \\
dx &= 5\cos(u)\,du
\end{aligned}$$

Then, $\sqrt{25-x^2} = \sqrt{25 - 25\sin^2(u)} = 5\cos(u)$, so

$$\begin{aligned}
\int \frac{x^3}{\sqrt{25-x^2}}\,dx &= \int \frac{125\sin^3(u)}{5\cos(u)} 5\cos(u)\,du \\
&= 125 \int \sin^3(u)\,du \\
&= 125 \int \left(1-\cos^2(u)\right)\sin(u)\,du.
\end{aligned}$$

Now we can make the substitution

$$\begin{aligned}
z &= \cos(u) \\
dz &= -\sin(u)du
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{25-x^2}}\,dx &= 125 \int \left(1-\cos^2(u)\right)\sin(u)\,du \\
&= -125 \int \left(1-z^2\right)\,dz \\
&= -125\left(z - \frac{z^3}{3}\right) + C \\
&= -125\left(\cos(u) - \frac{\cos^3(u)}{3}\right) + C
\end{aligned}$$

Normally, we would have to draw a triangle to express $\cos(u)$ in terms of $\sin(u)$, but we know that $\cos(u) = \sqrt{1-\sin^2(u)} = \sqrt{1-x^2}$, so

$$\begin{aligned}
\int \frac{x^3}{\sqrt{25-x^2}}\,dx &= -125\left(\cos(u) - \frac{\cos^3(u)}{3}\right) + C \\
&= -125\left((1-x^2)^{1/2} - \frac{(1-x^2)^{3/2}}{3}\right) + C
\end{aligned}$$

(b) (9 points)

$$\int \frac{dx}{x^2\sqrt{x^2+4}}$$

We make the substitution

$$\begin{aligned}
x &= 2\tan(u) \\
dx &= 2\sec^2(u)\,du
\end{aligned}$$

Then, since $\tan^2(u) + 1 = \sec^2(u)$,

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2(u)\,du}{4\tan^2(u)\sqrt{4\tan^2(u)+4}}$$

$$\begin{aligned}
&= \int \frac{2\sec^2(u)\,du}{4\tan^2(u)2\sec(u)} \\
&= \frac{1}{4}\int \frac{\sec(u)\,du}{\tan^2(u)} \\
&= \frac{1}{4}\int \frac{1/\cos(u)}{\sin^2(u)/\cos^2(u)}\,du \\
&= \frac{1}{4}\int \frac{\cos(u)}{\sin^2(u)}\,du
\end{aligned}$$

Now we can make the substitution

$$\begin{aligned}
z &= \sin(u) \\
dz &= \cos(u)\,du
\end{aligned}$$

to get

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+4}} &= \frac{1}{4}\int \frac{\cos(u)}{\sin^2(u)}\,du \\
&= \frac{1}{4}\int z^{-2}\,dz \\
&= -\frac{1}{4}z^{-1} + C \\
&= -\frac{1}{4z} + C \\
&= -\frac{1}{4\sin(u)} + C
\end{aligned}$$

Now we have to draw a triangle. We find that

$$\sin(u) = \frac{x}{\sqrt{x^2+4}}$$

so

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{1}{4\sin(u)} + C = -\frac{\sqrt{x^2+4}}{4x} + C.$$

12. (36 pts) Solve the following integrals.

(a) (9 points)

$$\int \frac{2}{x^2-x-6}\,dx$$

We use partial fractions to get

$$\frac{2}{x^2-x-6} = \frac{2}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{x+2}.$$

Solving for A and B, we get

$$2 = A(x+2) + B(x-3)$$

Setting $x = 3$, we find $2 = 5A$, or $A = 2/5$. With $x = -2$, we get $2 = -5B$, so $B = -2/5$. Thus, doing a simple u-substitution in our heads, we get

$$\begin{aligned} \int \frac{2}{x^2 - x - 6}\, dx &= \frac{2}{5}\int \left(\frac{1}{x-3} - \frac{1}{x+2}\right) dx \\ &= \frac{2}{5}\left(\ln|x-3| - \ln|x+2|\right) + C \end{aligned}$$

(b) (9 points)

$$\int \frac{x^2 - x + 2}{x+1}\, dx$$

Using long division of polynomials, we find that

$$\frac{x^2 - x + 2}{x+1} = -2 + x + \frac{4}{1+x}$$

Integrating these terms, we find that

$$\int \frac{x^2 - x + 2}{x+1}\, dx = -2x + \frac{x^2}{2} + 4\ln|1+x| + C.$$

(c) (9 points)

$$\int \frac{x+4}{x^3 + 2x^2}\, dx$$

First,

$$x^3 + 2x^2 = x^2(x+2).$$

Using partial fractions, we must solve

$$\frac{x+4}{x^3+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

This leads to the equation

$$x + 4 = Ax(x+2) + B(x+2) + Cx^2$$

Setting $x = 0$, we find that $4 = 2B$, so $B = 2$. Next, letting $x = -2$, we get $2 = 4C$ so that $C = 1/2$. With $x = -1$, we get

$$3 = -A + B + C = -A + 2 + \frac{1}{2} = -A + \frac{5}{2}.$$

This implies that $A = -1/2$. Thus,

$$\begin{aligned}\int \frac{x+4}{x^3+2x^2}\,dx &= \int \left(-\frac{1}{2x} + \frac{2}{x^2} + \frac{1}{2}\cdot\frac{1}{x+2}\right) dx \\ &= -\frac{1}{2}\ln|x| - \frac{2}{x} + \frac{1}{2}\ln|x+2| + C.\end{aligned}$$

(d) (9 points)

$$\int \frac{1}{x^3+3x^2+2x}\,dx$$

Factoring the denominator, we get

$$x^3+3x^2+2x = x(x^2+3x+2) = x(x+1)(x+2).$$

Thus, we must solve

$$\frac{1}{x^3+3x^2+2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

We get the equation

$$1 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1).$$

Setting $x = 0$, we get $1 = 2A$, so $A = 1/2$. Setting $x = -1$, we get $1 = -B$, so $B = -1$. Finally, setting $x = -2$, we get $1 = 2C$, so that $C = 1/2$. Thus,

$$\begin{aligned}\int \frac{1}{x^3+3x^2+2x}\,dx &= \frac{1}{2}\int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x+2}\,dx \\ &= \frac{1}{2}\ln|x| - \ln|x+1| + \frac{1}{2}\ln|x+2| + C.\end{aligned}$$

13. (8 pts) Approximate

$$\int_0^4 \sqrt{x^3+1}\,dx$$

using 4 intervals of equal length, using the trapezoidal rule. You do not have to evaluate the square roots. $[0,4]$ will be divided into 4 intervals of length 1, so that $\Delta x = 1$. The endpoints will be 0,1,2,3,4. The trapezoidal rule says that

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\Big].$$

So, in our case, since the points are $0,1,2,3,4$, we get

$$\int_0^4 \sqrt{x^3+1}\,dx \approx \frac{1}{2}\left[1 + 2\sqrt{2} + 2\sqrt{9} + 2\sqrt{28} + \sqrt{65}\right].$$

14. (16 pts) Solve the following integrals.

(a) (8 points)

$$\int_{-1}^{4} \frac{1}{(x-1)^4}\,dx$$

Since the integrand blows up at $x = 1$, we must separate the integral into 2 pieces:

$$\begin{aligned}
\int_{-1}^{4} \frac{1}{(x-1)^4}\,dx &= \int_{-1}^{1} \frac{1}{(x-1)^4}\,dx + \int_{1}^{4} \frac{1}{(x-1)^4}\,dx \\
&= \int_{-1}^{1} (x-1)^{-4}\,dx + \int_{1}^{4} (x-1)^{-4}\,dx \\
&= -\frac{1}{3}(x-1)^{-3}\Big|_{-1}^{1-} + -\frac{1}{3}(x-1)^{-3}\Big|_{1+}^{4} \\
&= +\infty.
\end{aligned}$$

(b) (8 points)

$$\int_{-3}^{-1} \frac{1}{(x+2)^{1/3}}\,dx$$

Since the integrand blows up at $x = -2$, we must separate the integral into 2 pieces:

$$\begin{aligned}
\int_{-3}^{-1} \frac{1}{(x+2)^{1/3}}\,dx &= \int_{-3}^{-2} \frac{1}{(x+2)^{1/3}}\,dx + \int_{-2}^{-1} \frac{1}{(x+2)^{1/3}}\,dx \\
&= \int_{-3}^{-2} (x+2)^{-1/3}\,dx + \int_{-2}^{-1} (x+2)^{-1/3}\,dx \\
&= \frac{3}{2}(x+2)^{2/3}\Big|_{-3}^{-2} + \frac{3}{2}(x+2)^{2/3}\Big|_{-2}^{-1} \\
&= 0 - \frac{3}{2}(-1)^{2/3} + \frac{3}{2}\cdot 1^{2/3} + 0 \\
&= -\frac{3}{2} + \frac{3}{2} \\
&= 0
\end{aligned}$$

15. (8 pts) For the following problem, **SET UP THE INTEGRAL, BUT DO NOT SOLVE IT**. What is the length of the curve $y = x^3 + x$ between $x = -1$ and $x = 3$? For $y = f(x)$ between $x = a$ and $x = b$, the length of the curve is

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

In our case,

$$f'(x) = 3x^2 + 1$$

so the length of the curve would be

$$\int_{-1}^{3} \sqrt{1 + (3x^2+1)^2}\,dx \quad = \quad \int_{-1}^{3} \sqrt{9x^4 + 6x^2 + 2}\,dx.$$

16. (10 pts) Suppose that a cubical tank with 2 meters on each side is full of water. Find the force on one of the vertical sides, in Newtons.

We have to do an integral, since different parts of the side feel different pressures. Let x be the depth. The slice of a side with depth between x and $x + \Delta x$ will have area $2\Delta x$. Since the depth will be x, the force on that slice will be

$$A\rho g d = 2\Delta x \cdot 1000 \cdot 9.8x \text{ Newtons} = 19600x\Delta x \text{ Newtons}$$

where A is the area of the slice, ρ is the density of water, g is the gravitational constant, and d is the depth. We must add together the force on all of the slices, so we must solve the following integral.

$$\int_0^2 19600x\,dx = 19600\frac{x^2}{2}\Big|_0^2 = 39200$$

So, the answer is 39200 Newtons.

17. (10 pts) Find the center of mass of a right triangle with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$. The triangle is bounded by the lines $x = 0$, $y = 0$, $x = 1$, and $y = 1 - x$. The center of mass in the x-variable is given by

$$\frac{\int_0^1 xf(x)dx}{\int_0^1 f(x)dx}$$

Now

$$\int_0^1 f(x)dx = \int_0^1 (1 - x)dx = \frac{1}{2}$$

and

$$\int_0^1 xf(x)dx = \int_0^1 x(1 - x)dx = \int_0^1 (x - x^2)dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Thus, the center of mass in the x-variable is

$$\frac{1/6}{1/2} = \frac{1}{3}.$$

By symmetry, the answer is the same for the center of mass in the y-variable. We can also find the answer by the formula,

$$\frac{(1/2)\int_0^1 f(x)^2dx}{\int_0^1 f(x)dx}.$$

Now the denominator is the same as before, $1/2$. As for the numerator,

$$\frac{1}{2}\int_0^1 f(x)^2dx = \frac{1}{2}\int_0^1 (1 - x)^2dx = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

which is the same as for the x-variable. Therefore, the center of mass for the y-variable will also by $1/3$. The total center of mass is at the point

$$(x, y) = \left(\frac{1}{3}, \frac{1}{3}\right).$$