142s24midterm2_fin

Math 142: Calculus II

Midterm II

April 2, 2024

Name: ______________________________ (Please print clearly)

UR ID: ______________________________

Instructions:

- You have 75 minutes to work on this exam. You are responsible for checking that this exam has all 10 pages. **Please do not remove any pages.**
- No calculators, phones, electronic devices, books, or notes are allowed during the exam, except for the provided formula sheet.
- **Show all work and justify all answers.** Correct answers with insufficient work will not be given full credit.
- Numerical or algebraic simplifications of answers are not required, **except when specifically stated otherwise.**
- A blank page for scratch work is provided at the end of the exam. **Work on this page will not be graded.** Please show your work on the page containing the relevant question.
- Clearly circle all final answers.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

FOR REFERENCE, NO QUESTION ON THIS PAGE

Table of common antiderivatives:

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec^2 x$	$\tan x$
$x^n (n \neq 1)$	$\frac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\frac{1}{1+x^2}$	$\tan^{-1} x$

Note that the following formulas may not be applicable in every situation, and may need to be manipulated to suit individual problems.

Volume by disks/washers:

$$V = \int_a^b \pi(R^2 - r^2)dx \quad \text{or} \quad V = \int_a^b \pi(R^2 - r^2)dy$$

Volume by cylindrical shells:

$$V = \int_a^b 2\pi rh dx \quad \text{or} \quad V = \int_a^b 2\pi rh dy$$

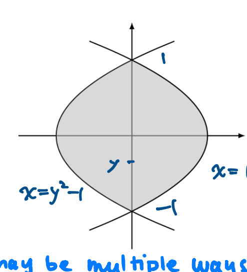
Work: The work done to move an object along a line from a to b by force $f(x)$ is

$$W = \int_a^b f(x)dx.$$

Hooke's Law: The force required to hold a spring distance x beyond its natural length is

$$F = kx.$$

1. (15 points) Consider the region bounded by the curves $x = 1 - y^2$ and $x = y^2 - 1$. This region is plotted below.



$\begin{cases} x = 1-y^2 \\ x = y^2 - 1 \end{cases}$

$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

→Note: there may be multiple ways to explain your choice

(a) Which variable of integration would you choose to express the area of this region as a single integral? Explain your choice.

Area $= \int_{-1}^{1} (1-y^2) - (y^2-1)\, dy$, i.e. w.r.t. y:

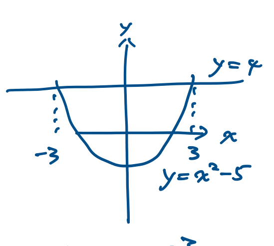
this is because the region has a single left and right boundary curve, but not a single top/bottom curve

(b) Find the area of the region.

Area $= \int_{-1}^{1} 2 - 2y^2\, dy = \left[2y - \frac{2}{3}y^3\right]_{-1}^{1}$

$= \frac{8}{3}$

2. (10 points)
Find the area of the region bounded by the curves $y = 4$ and $y = x^2 - 5$.



$\begin{cases} y = 4 \\ y = x^2 - 5 \end{cases} \Rightarrow x = \pm 3$

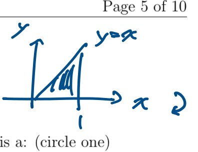
Area $= \int_{-3}^{3} 4 - (x^2 - 5)\, dx$

$= \int_{-3}^{3} 9 - x^2\, dx = \left(9x - \frac{1}{3}x^3\right)\Big|_{-3}^{3}$

$= 9 \cdot 6 - \frac{1}{3} \cdot 2 \cdot 27 = 36$

3. (15 points)
Consider the region bounded by the curves

$$y = x, \quad y = 0, \quad x = 1.$$



(a) The solid obtained by rotating this region about the x-axis is a: (circle one)

Circle Cone Sphere

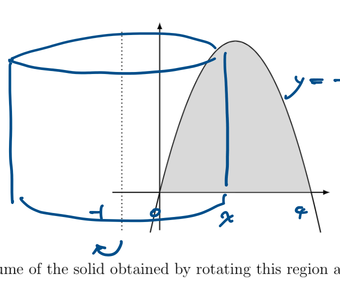
(b) Using the **washer method**, find the volume of the solid obtained by rotating this region about the x-axis.

$\int_0^1 \pi x^2 dx = \pi \cdot \frac{1}{3} x^3 \Big|_0^1$

$= \frac{\pi}{3}$

4. (15 points) Consider the region bounded by the following curves:

$$y = -x^2 + 4x, \quad y = 0.$$



$y = -x^2 + 4x$

$0 = x(-x+4)$

$x = 0$ or 4

Find the volume of the solid obtained by rotating this region about the line $x = -1$.

Volume $= \int_0^4 2\pi (x+1)(-x^2+4x)\, dx$

$= 2\pi \int_0^4 -x^3 + \underbrace{4x^2 - x^2}_{3x^2} + 4x\, dx$

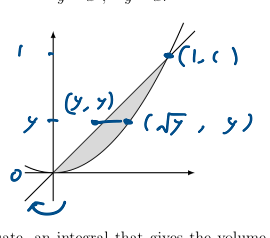
$= 2\pi \left(-\frac{1}{4}x^4 + x^3 + 2x^2\right)\Big|_0^4$

$= 2\pi(-4^3 + 4^3 + 32)$

$= 64\pi$

5. (10 points) Consider the solid obtained by rotating the region bounded by the following curves about the y-axis:

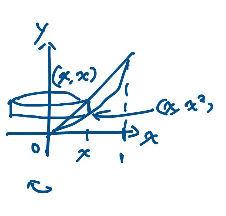
$$y = x^2, \quad y = x.$$



(a) Set up, but do not evaluate, an integral that gives the volume of this solid using the washer method.

$\int_0^1 \pi\left((\sqrt{y})^2 - y^2\right) dy$

(b) Set up, but do not evaluate, an integral that gives the volume of this solid using the method of cylindrical shells.



$\int_0^1 2\pi x(x - x^2)\, dx$

6. (15 points) Consider a spring with a natural length of 50 cm. Suppose that 2 J of work is required to stretch the spring from 50 cm to 75 cm.

0.5 0.5 0.75

(a) Determine the spring constant k.

$\int_0^{0.25} kx\, dx = 2$

$\Rightarrow k \cdot \frac{1}{2}\left(\frac{1}{4}\right)^2 = 2 \Rightarrow k = 64$

(b) Find the work required to stretch the spring from 75 cm to 100 cm.

0.75 1

Work $= \int_{0.75-0.5}^{1-0.5} 64x\, dx$

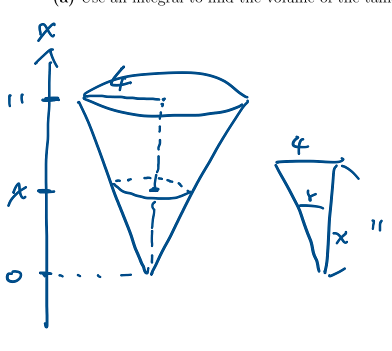
$= 64 \cdot \frac{1}{2} x^2 \Big|_{\frac{1}{4}}^{\frac{1}{2}}$

$= 32\left(\frac{1}{4} - \frac{1}{16}\right) = 32 \cdot \frac{3}{16}$

$= 6$ J

7. (15 points) A tank has the shape of an upside-down (i.e., with the point facing down) right circular cone with height 11 ft and base radius 4 ft.

(a) Use an integral to find the volume of the tank.



$\frac{r}{x} = \frac{4}{11} \Rightarrow r = \frac{4}{11}x$

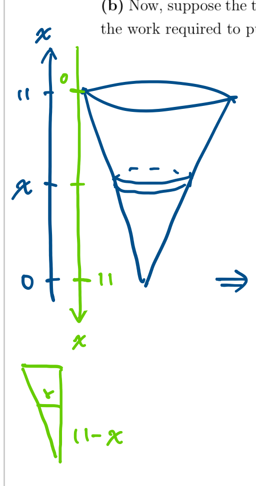
$\int_0^{11} \pi\left(\frac{4}{11}x\right)^2 dx$

$= \frac{16}{121}\pi \cdot \frac{1}{3} \cdot 11^3$

$= \frac{16 \cdot 11}{3}\pi$

$= \frac{176}{3}\pi$ ft³

(b) Now, suppose the tank is filled to the top with a liquid that weighs 50 lbs/ft^3. Determine the work required to pump all the liquid out over the top edge of the tank.



volume of x-layer $= \frac{16}{121}\pi x^2 \Delta x$ ft³

weight $= \frac{16}{121}\pi x^2 \cdot 50\, \Delta x$ lb

work of pumping x-layer

$= \frac{800}{121}\pi x^2 (11 - x)\Delta x$ ft-lb

$\Rightarrow$ Total work $= \int_0^{11} \frac{800}{121}\pi\, x^2(11-x)\, dx$ $\boxed{50\left(\frac{4}{11}x\right)^2\pi(11-x)}$

$= \frac{800}{121}\pi \int_0^{11} 11x^2 - x^3\, dx$

$= \frac{800}{121}\pi\left(\frac{11}{3} \cdot 11^3 - \frac{1}{4} \cdot 11^4\right)$ ft-lb

$= 800 \times 121 \times \frac{1}{12}\pi$ ft-lb

$= \frac{200 \times 121}{3}\pi$ ft-lb $= \frac{24200}{3}$ ft-lb

$\frac{r}{11-x} = \frac{4}{11}$

$\Rightarrow r = \frac{4}{11}(11-x)$

$\boxed{\int_0^{11} 50\pi\left(\frac{4}{11}\right)^2 (11-x)^2 x\, dx} = 50\pi \cdot \frac{16}{121}\left(\frac{121}{2} \cdot 11^2 - \frac{22}{3} 11^3 + \frac{11^4}{4}\right)$

(scratchwork page)