Math 142: Midterm 1

University of Rochester

October 4, 2022

Name: Solutions

UR ID: ____________________

UR E-mail: ____________________

Section	"X" your class time
MW 9 AM	
MW 3:25 PM	

- You are allowed one page, single-sided of notes. No other resources are permitted.
- The exam questions are on pages 2-11 of this packet.
- Each part of each question is on its own page. All work you want graded for that problem should be contained entirely on that page, unless:
- If you need more space on a problem, use the **Scratch work** pages at the end of the exam, and make sure to make a note on the problem page that you are doing so.
- **Do not tear off the scratch work pages.**
- Copy and sign the Honor Pledge: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

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Signature: ____________________

Question:	1	2	3	4	5	6	Total
Points:	30	10	15	15	15	15	100

1. Consider the function $f(x)$ defined by

$$f(x) = \frac{x}{x^2 - 1}.$$

The first and second derivatives of $f(x)$ are

$$f'(x) = -\frac{1 + x^2}{(x^2 - 1)^2} \quad \text{and} \quad f''(x) = \frac{\overset{2x(3+x^2)}{\cancel{2x(3+x^2)}}}{(x^2 - 1)^3}.$$

(a) (2 points) What is the domain of $f(x)$?

ANSWER:

$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(b) (2 points) List the x-intercepts of $f(x)$.

$f(x) = 0 \iff x = 0$

ANSWER:

$x = 0$

(c) (2 points) List the y-intercept of $f(x)$.

$f(0) = 0$

ANSWER:

$y = 0$

(d) (2 points) Find all the vertical asymptotes of $f(x)$, or explain why none exist.

$$\lim_{x \to 1^+} \frac{x}{x^2-1} = +\infty, \quad \lim_{x \to -1^-} \frac{x}{x^2-1} = -\infty$$

ANSWER:

$x = \pm 1$

(e) (2 points) Find all the horizontal asymptotes of $f(x)$, or explain why they do not exist.

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0$$

ANSWER:

$y = 0$ in both directions

(f) (2 points) Find all the intervals where $f(x)$ is **increasing**.

$$f'(x) > 0 \iff \underbrace{-\frac{1+x^2}{(x^2-1)^2} > 0}_{\text{no } \forall x \text{ exist}}$$

such

ANSWER:

None

(g) (2 points) Find all the intervals where $f(x)$ is **decreasing**.

ANSWER:

$(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$

(h) (2 points) Find all the critical numbers of $f(x)$, or explain why none exist.

ANSWER:

None: $f'(x) \neq 0$ for all x, $f'(x)$ DNE only at ± 1 (not in domain of f) no change in sign chart

(i) (2 points) Find all the intervals where $f(x)$ is **concave up.**

$$f''(x) > 0 \iff \frac{2x(3+x^2)}{(x^2-1)^3} > 0$$

$$\iff \frac{2x}{(x^2-1)^3} > 0$$

f'': $-$ | $+$ | $-$ | $+$ at -1, 0, 1

ANSWER:

$(-1, 0)$ & $(1, \infty)$

(j) (2 points) Find all the intervals where $f(x)$ is **concave down.**

ANSWER:

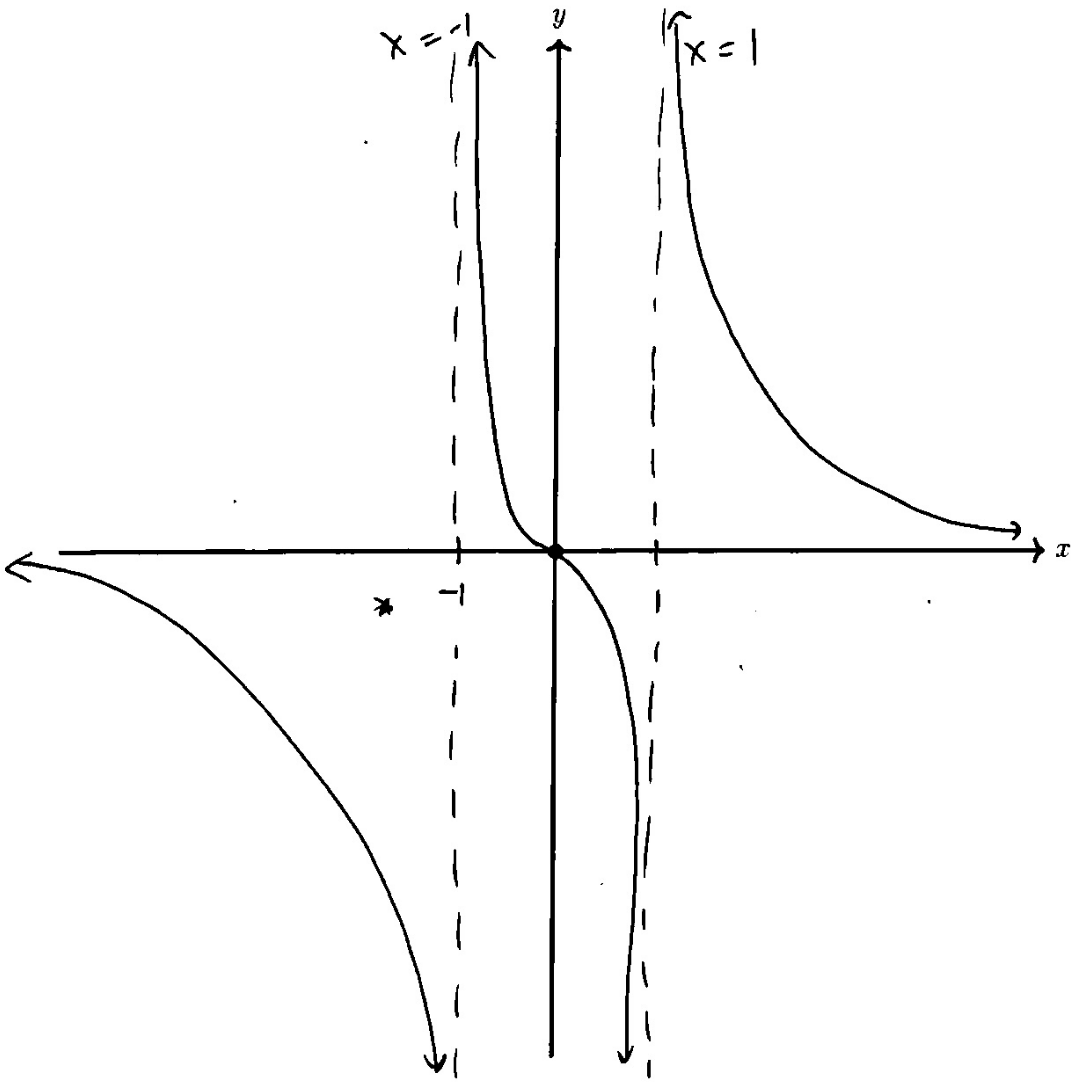
$(-\infty, -1)$ & $(0, 1)$

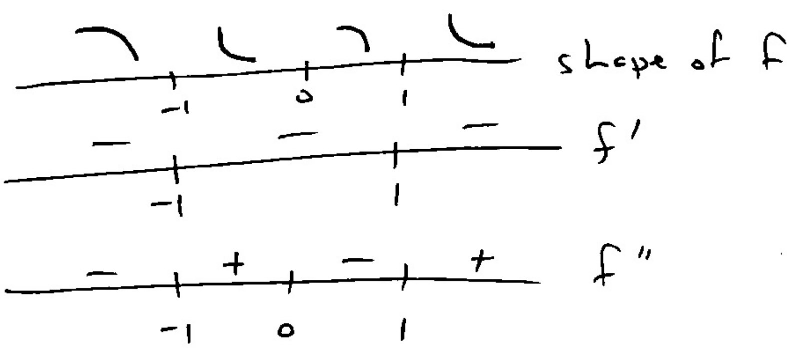
(k) (2 points) Find all the inflection points of $f(x)$, or explain why none exist.

ANSWER:

$(0, 0)$

(l) (8 points) Use your work from parts (a)-(k) to graph $f(x)$ below. Note that you may scale the axes how you like (i.e. prioritize a good sketch over using 1 tick mark to represent 1 unit along an axis).





2. (10 points) Find the absolute minimum and maximum values of $f(x) = x^4 - 2x^2$ on the interval $[-1, 2]$.

use Closed Interval Method

$f'(x) = 4x^3 - 4x$ ← defined everywhere

$= 0 \iff 4x(x^2-1) = 0$

$\iff 4x(x-1)(x+1) = 0$

$\iff x = 0,\ x = 1,\ x = -1$

$f(-1) = (-1)^4 - 2(-1)^2 = 1 - 2 = -1$

$f(0) = 0$

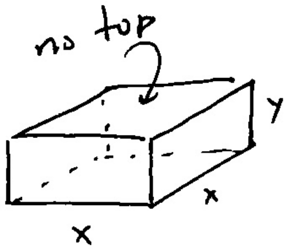
$f(1) = 1^4 - 2 \cdot 1^2 = 1 - 2 = -1$

$f(2) = \cancel{f}\ 2^4 - 2 \cdot 2^2 = 16 - 8 = 8$

ANSWER:

Abs max: 8, abs min: -1

3. (15 points) If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest volume of the box. Make sure to completely justify your answer.



$V = x^2 y$ ← maximize

Know: $x^2 + 4xy = 1200$

$\Rightarrow y = \dfrac{1200 - x^2}{4x} = \dfrac{300}{x} - \dfrac{1}{4}x$

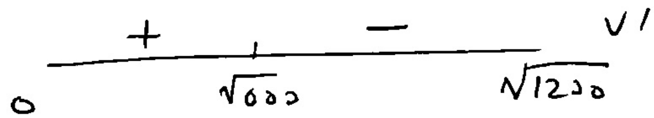
$$\Rightarrow V = x^2\left(\frac{300}{x} - \frac{x}{4}\right) = \underbrace{300x - \frac{x^3}{4}}_{\text{maximize}}$$

~~maximize on $(0,\infty)$~~

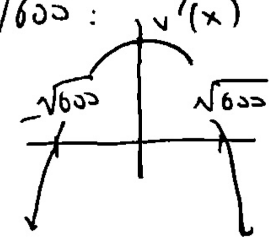
Domain need $0 < x < \sqrt{1200}$: x in $(0, \sqrt{1200})$

$V' = 300 - \dfrac{x^2}{2} = 0 \iff \dfrac{x^2}{2} = 300 \iff x^2 = 600$

$\iff x = \pm\sqrt{600}$: only crit in domain is $x = \sqrt{600}$



$V'(x)$ is quadratic w/ roots $\pm\sqrt{600}$:



V maximized when $x = \sqrt{600}$

ANSWER: $V(\sqrt{600}) = 300\sqrt{600} - \dfrac{(\sqrt{600})^3}{4}$

$$\boxed{300\sqrt{600} - \frac{(\sqrt{600})^3}{4}}$$

4. Compute the following indefinite integrals:

(a) (5 points) $\int \frac{3}{x^2} + e^x + \sec^2 x \, dx$

ANSWER:

$$\boxed{-\frac{3}{x} + e^x + \tan(x) + C}$$

(b) (5 points) $\int \left(x + \frac{1}{x}\right)(2x+1) \, dx$

$$= \int 2x^2 + x + 2 + \frac{1}{x} \, dx$$

ANSWER:

$$\boxed{\frac{2}{3}x^3 + \frac{x^2}{2} + 2x + \ln|x| + C}$$

(c) (5 points) $\int \frac{2}{1+x^2} + \frac{1+x^2}{x^2}\, dx$

$$= \int 2 \cdot \frac{1}{1+x^2} + \frac{1}{x^2} + \cancel{\frac{x^2}{x^2}}^{=1}\, dx$$

ANSWER:

$$\boxed{2\arctan(x) - \frac{1}{x} + x + C}$$

5. (15 points) Consider a particle on the x-axis which starts to move from the origin at $t = 0$, i.e. if $x(t)$ is the function indicating the location of the particle at time t, then $x(0) = 0$. If the velocity of this particle at time t is given by the function $v(t) = t^2 + t + e^t$, find the location of the particle at $t = 2$.

$$v(t) = t^2 + t + e^t$$

$$\Rightarrow \underbrace{p(t)}_{\text{position fct}} = \frac{t^3}{3} + \frac{t^2}{2} + e^t + C$$

$$p(0) = 0 \Rightarrow \frac{0^3}{3} + \frac{0^2}{2} + e^0 + C = 0$$

$$\Rightarrow 1 + C = 0$$

$$\Rightarrow C = -1$$

$$p(2) = \frac{2^3}{3} + \frac{2^2}{2} + e^2 - 1$$

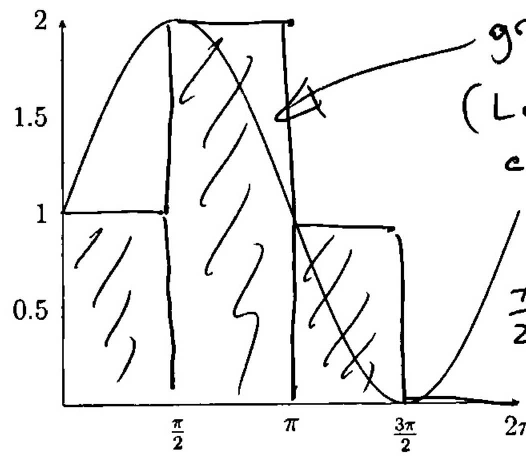
$$= \frac{8}{3} + 2 + e^2 - 1$$

$$= \frac{11}{3} + e^2$$

ANSWER:

$$\frac{11}{3} + e^2$$

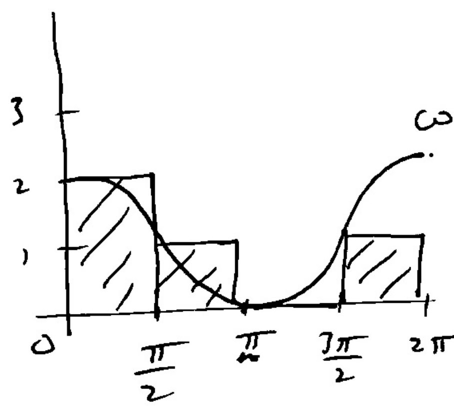
6. (15 points) Find the left endpoint Riemann sum L_4 (i.e. using 4 subintervals) for the function $f(x) = \cos(x) + 1$ above the interval $[0, 2\pi]$. **Simplify your answer as much as possible.** A graph of $f(x)$ is given below.



graph of $\sin(x)+1$:(

(L_4 for this fct also accepted :))

L_4 for $\sin(x)+1$: $\frac{\pi}{2}(1+2+1+0) = 2\pi$ also



$$L_4 = \left(f(0) + f\left(\tfrac{\pi}{2}\right) + f(\pi) + f\left(\tfrac{3\pi}{2}\right)\right)\frac{\pi}{2}$$

$$= (2 + 1 + 0 + 1) \cdot \frac{\pi}{2}$$

$$= 4 \cdot \frac{\pi}{2} = 2\pi$$

ANSWER:

2π