Math 142: Final Exam

University of Rochester

December 18, 2022

Name: Solutions

UR ID: ______________________

UR E-mail: ______________________

Section	"X" your class time
MW 9 AM	
MW 3:25 PM	

- You are allowed one page, single-sided of notes. No other resources are permitted.
- The exam questions are on pages 2-21 of this packet. **Part A consists of problems 1-6. Part B consists of problems 7-11.**
- Each part of each question is on its own page. All work you want graded for that problem should be contained entirely on that page, unless:
- If you need more space on a problem, use the **Scratch work** pages at the end of the exam, and make sure to make a note on the problem page that you are doing so.
- **Do not tear off the scratch work pages.**
- Copy and sign the Honor Pledge: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.*

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Signature: ______________________

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	25	10	20	15	10	20	30	20	10	30	10	200

Part A

1. Consider the following function and its derivatives:

$$f(x) = \frac{1}{1-x^2} \qquad f'(x) = \frac{2x}{(1-x^2)^2} \qquad f''(x) = -\frac{2(3x^2+1)}{(x^2-1)^3}.$$

(a) (2 points) What is the domain of $f(x)$? Write it in interval notation.
ANSWER:

$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(b) (2 points) What are the x and y intercepts of $f(x)$ (if any)?
ANSWER:

No x-intercepts; y-intercept is $f(0) = \frac{1}{1} = 1$.

(c) (2 points) What are the horizontal and vertical asymptotes of $f(x)$ (if any)?
ANSWER:

HA @ $x = \pm 1$, VA $y = 0$ in both directions

(d) (3 points) On which intervals is $f(x)$ increasing, and on which intervals is $f(x)$ decreasing?
ANSWER:

$-$ -1 $-$ 0 $+$ 1 $+$ f'

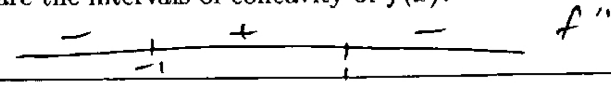
Increasing: $(0,1)$ & $(1,\infty)$
Decreasing: $(-\infty,-1)$ & $(-1,0)$

(e) (3 points) What are the local extrema of $f(x)$, if any?
ANSWER:

Local min @ $x = 0$.

$(f(0) = 1)$

(f) (3 points) What are the intervals of concavity of $f(x)$?
ANSWER:

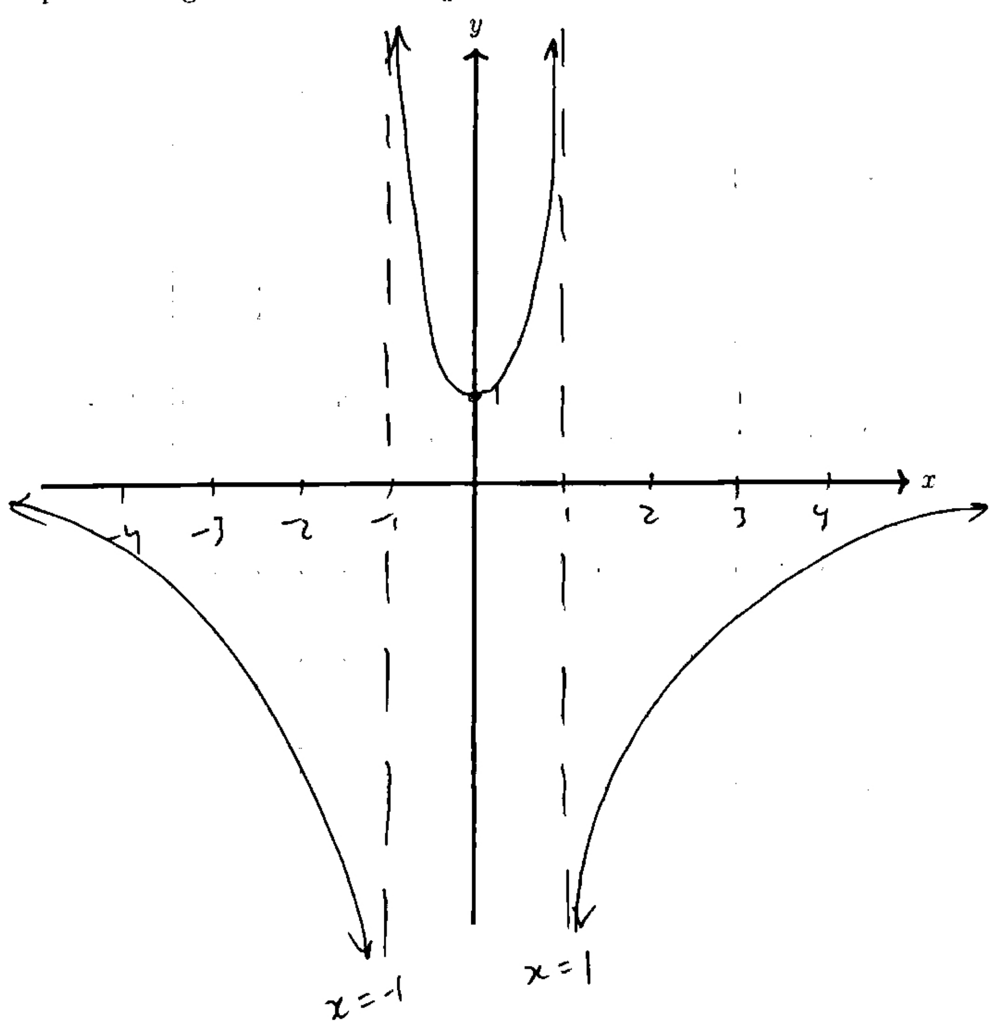


Concave up: $(-1, 1)$
Concave down: $(-\infty, -1)$, $(1, \infty)$

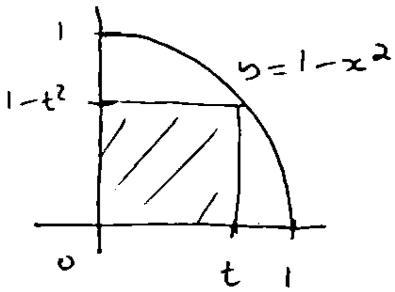
(g) (3 points) What are the inflection points of $f(x)$, if any?
ANSWER:

None

(h) (7 points) Sketch the graph of $f(x)$. You may scale the axes how you like (i.e. prioritize a good sketch over using 1 tick mark to represent 1 unit along an axis).



2. (10 points) What is the area of the largest rectangle in the first quadrant of the xy-plane with one corner at the origin and opposite corner on the graph of $f(x) = 1 - x^2$?



$A(t) = t(1-t^2)$

for $0 < t < 1$ (domain $(0,1)$)

$A(t) = t - t^3$

$A'(t) = 1 - 3t^2 = 0 \Leftrightarrow 3t^2 = 1$

$\Leftrightarrow t^2 = \frac{1}{3} \Leftrightarrow t = \pm 1/\sqrt{3}$

only include $t = \frac{1}{\sqrt{3}}$ (other CN not in dom)

$\begin{array}{cccc} & + & & - \\ 0 & & \frac{1}{\sqrt{3}} & & 1 \end{array}$ A' $\Rightarrow A$ has an abs. max @ $t = \frac{1}{\sqrt{3}}$ in $(0,1)$.

$$A\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(1 - \left(\frac{1}{\sqrt{3}}\right)^2\right)$$

$$= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{\sqrt{3}}{3} \cdot \frac{2}{3} = \frac{2\sqrt{3}}{9}$$

ANSWER:

$$\boxed{\frac{2\sqrt{3}}{9}}$$

3. Consider a particle moving on the real line, whose acceleration as a function of time is given by $a(t) = \dfrac{2}{(t+1)^3}$ for $t \geq 0$. Moreover, assume that the initial position and velocity of the particle are given by $x(0) = 0$ and $v(0) = -1$, respectively.

(a) (10 points) What is the location of the particle at time $t = 1$?

$$\int a(t)\,dt = \int \frac{2}{(t+1)^3}\,dt = \int 2(t+1)^{-3}\,dt$$

$$= -(t+1)^{-2} + C \leftarrow v(t)$$

$$v(0) = -1 \Rightarrow -1 = -(0+1)^{-2} + C$$

$$= -1 + C$$

$$\Rightarrow C = 0$$

$$\Rightarrow v(t) = -(t+1)^{-2} = -\frac{1}{(t+1)^2}$$

$$\int v(t)\,dt = \int -(t+1)^{-2}\,dt = (t+1)^{-1} + C \leftarrow x(t)$$

$$x(0) = 0 \Rightarrow 0 = (0+1)^{-1} + C$$

$$= 1 + C$$

$$\Rightarrow C = -1$$

$$\Rightarrow x(t) = (t+1)^{-1} - 1 = \frac{1}{t+1} - 1$$

$$x(1) = \frac{1}{2} - 1 = -\frac{1}{2}$$

ANSWER:

$$\boxed{-\frac{1}{2}}$$

(b) (10 points) What is the velocity of the particle at time $t = 1$?

$$v(1) = -\frac{1}{2^2} = -\frac{1}{4}$$

ANSWER:

$$\boxed{-\frac{1}{4}}$$

4. Compute the following integrals.

(a) (5 points) $\int 2xe^{x^2+1}\,dx$

$u = x^2+1$, $du = 2x\,dx$

$= \int e^u \, du$

$= e^u + C$

$= e^{x^2+1} + C$

ANSWER:

$$\boxed{e^{x^2+1} + C}$$

(b) (5 points) $\int t^2\sqrt{t-1}\,dt$

$u = t-1$, $du = dt$

$t = u+1$

$$= \int (u+1)^2 \sqrt{u}\, du$$

$$= \int (u+1)^2 u^{1/2}\, du$$

$$= \int (u^2+2u+1)u^{1/2}\, du$$

$$= \int u^{5/2} + 2u^{3/2} + u^{1/2}\, du$$

$$= \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} \cancel{d} + C$$

ANSWER:

$$\boxed{\frac{2}{7}(t-1)^{7/2} + \frac{4}{5}(t-1)^{5/2} + \frac{2}{3}(t-1)^{3/2} + C}$$

(c) (5 points) $\int \frac{2\cos(x)}{1+\sin^2(x)}\,dx$

$u = \sin(x)$

$du = \cos(x)\,dx$

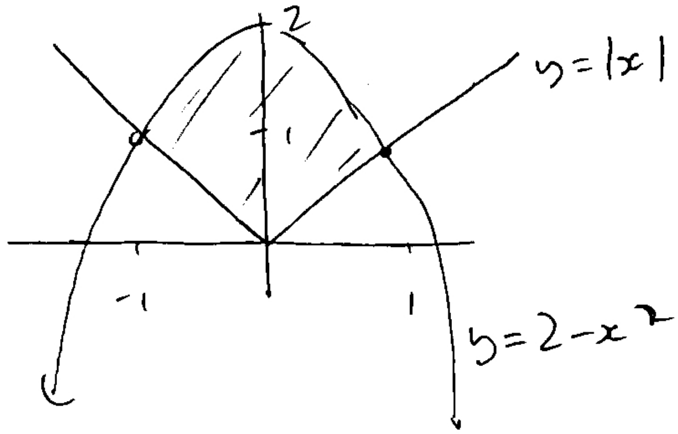
$= \int \frac{2}{1+u^2}\,du$

$= 2\arctan(u) + C$

ANSWER:

$2\arctan(\sin(x)) + C$

5. (10 points) Find the area enclosed by the graphs of $y = |x|$ and $y = 2 - x^2$.



$$\text{Area} = \int_{-1}^{1} (2 - x^2) - |x| \, dx$$

$$= 2\int_0^1 (2 - x^2) - x \, dx$$

$$= 2\left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1$$

$$= 2\left(2 - \frac{1}{3} - \frac{1}{2}\right)$$

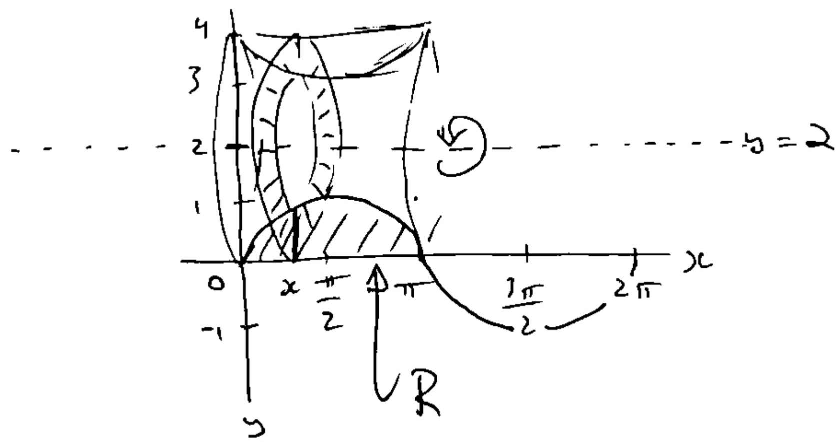
$$= 2\left(\frac{7}{6}\right) = \frac{7}{3}$$

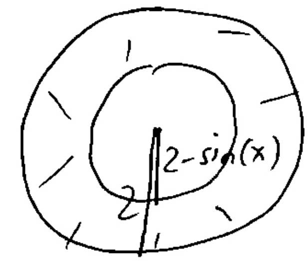
ANSWER:

$$\boxed{\frac{7}{3}}$$

6. Let R be the region in the plane above the x-axis and below the graph of $y = \sin(x)$ from $x = 0$ to $x = 2\pi$.

   (a) (10 points) Write down (**but do not calculate**) an integral expressing the volume of the solid region obtained by rotating R around the horizontal line $y = 2$.

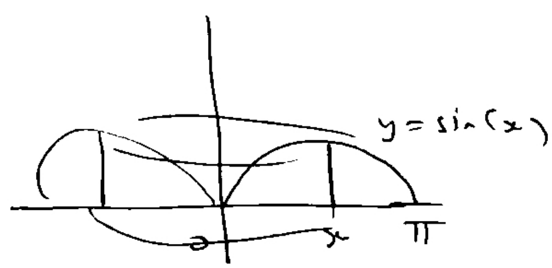




ANSWER:

$$\int_0^{\pi} \pi\left(4-(2-\sin(x))^2\right)dx$$

   (b) (10 points) Write down (**but do not calculate**) an integral expressing the volume of the solid region obtained by rotating R around the y-axis.



Shell method

ANSWER:

$$\int_0^{\pi} 2\pi x \sin(x)\,dx$$

Part B

7. Compute the following integrals.

(a) (10 points) $\int t^2 \ln(2t)\,dt$

$u = \ln(2t) \qquad dv = t^2\,dt$

$du = \frac{2}{2t}\,dt \qquad v = \frac{t^3}{3}$

$= \frac{1}{t}\,dt$

$$\int t^2 \ln(2t)\,dt = \frac{t^3 \ln(2t)}{3} - \int \frac{t^3}{3} \cdot \frac{1}{t}\,dt$$

$$= \frac{t^3 \ln(2t)}{3} - \int \frac{t^2}{3}\,dt$$

$$= \frac{t^3 \ln(2t)}{3} - \frac{t^3}{9} + C$$

ANSWER:

$$\boxed{\frac{t^3 \ln(2t)}{3} - \frac{t^3}{9} + C}$$

(b) (10 points) $\int \sin^3(x)\cos^2(x)\,dx$

$$= \int \sin^2(x)\cos^2(x)\sin(x)\,dx$$

$$= \int (1-\cos^2(x))\cos^2(x)\sin(x)\,dx$$

$u = \cos(x)$
$du = -\sin(x)\,dx$

$$\cancel{= -\int (\cos^2(x)-1)}$$

$$= \int (1-u^2)u^2(-du)$$

$$= \int (u^2-1)u^2\,du$$

$$= \int u^4 - u^2\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

ANSWER:

$$\boxed{\frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C}$$

(c) (10 points) $\int_0^1 x^2 e^{-x}\, dx$

$$u = x^2 \qquad dv = e^{-x}\, dx$$

$$du = 2x\, dx \qquad v = -e^{-x}$$

$$= -x^2 e^{-x}\Big|_0^1 - \int_0^1 (-e^{-x})\, 2x\, dx$$

$$= -x^2 e^{-x}\Big|_0^1 + 2\int_0^1 x e^{-x}\, dx$$

$$= -e^{-1} + \underbrace{2\int_0^1 x e^{-x}\, dx}$$

$$u = x \qquad dv = e^{-x}\, dx$$

$$du = dx \qquad v = -e^{-x}\, \cancel{dx}$$

$$= -\frac{1}{e} + 2\left(-x e^{-x}\Big|_0^1 - \int_0^1 (-e^{-x})\, dx\right)$$

$$= -\frac{1}{e} + 2\left(-\frac{1}{e} + \int_0^1 e^{-x}\, dx\right)$$

$$= -\frac{3}{e} + 2\int_0^1 e^{-x}\, dx$$

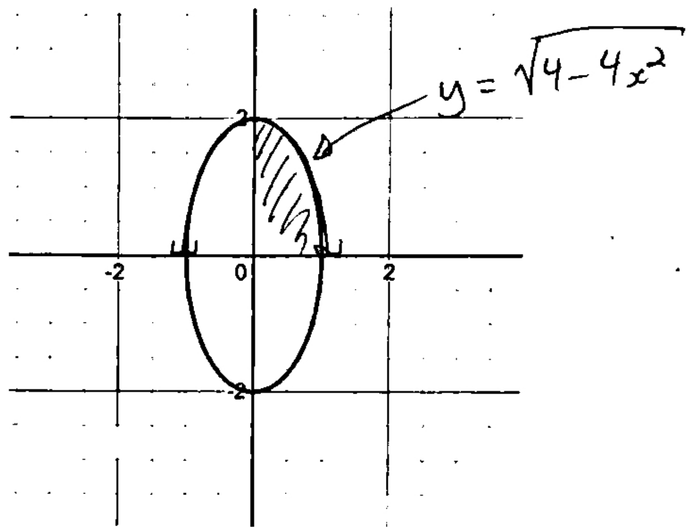
$$= -\frac{3}{e} - 2e^{-x}\Big|_0^1 = -\frac{3}{e} - 2\left(\frac{1}{e} - 1\right) = \cancel{\frac{4}{e}}$$

$$= 2 - \frac{5}{e}$$

ANSWER:

$$\boxed{\cancel{\frac{4}{e}}\ 2 - \frac{5}{e}}$$

8. Consider the ellipse $x^2 + \frac{y^2}{4} = 1$, pictured below. Let R be the first quadrant region enclosed by the ellipse and the coordinate axes.



(a) (10 points) Write the area A of R as an integral.

$$x^2 + \frac{y^2}{4} = 1 \implies \frac{y^2}{4} = 1 - x^2$$

$$\implies y^2 = 4 - 4x^2$$

$$\implies y = \pm\sqrt{4-4x^2}$$

upper half of ellipse : $y = \sqrt{4-4x^2}$

ANSWER:

$$\boxed{A = \int_0^1 \sqrt{4-4x^2}\,dx}$$

(b) (10 points) Use trigonometric substitution to compute A.

$$A = 4\int_0^1 \sqrt{4-4x^2}\,dx$$

$$\cancel{= 4\int_0^1 \sqrt{4-(2x}}$$

$x=0 \Rightarrow \theta = 0,\ x=1 \Rightarrow \theta = \frac{\pi}{2}$

$\Uparrow$

$$= 4\int_0^1 2\sqrt{1-x^2}\,dx$$

Let $x = \sin(\theta)$, so $dx = \cos(\theta)\,d\theta$ & $\sqrt{1-x^2} = \cos(\theta)$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2(\theta)\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} 1+\cos(2\theta)\,d\theta$$

$$= \left(\theta + \frac{\sin(2\theta)}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

ANSWER:

$$\boxed{\frac{\pi}{2}}$$

9. (10 points) Find $\int \frac{x^4}{x^3 - 2x^2 + x}\,dx$.

$$\begin{array}{r} x + 2 \\ x^3 - 2x^2 + x \,\big)\, \overline{x^4 \qquad\qquad\qquad} \\ -(x^4 - 2x^3 + x^2) \\ \hline 2x^3 - x^2 \\ -(2x^3 - 4x^2 + 2x) \\ \hline 3x^2 - 2x \end{array}$$

$$\int \frac{x^4}{x^3 - 2x^2 + x}\,dx = \int (x+2) + \frac{3x^2 - 2x}{x^3 - 2x^2 + x}\,dx$$

$$= \int (x+2) + \frac{3x - 2}{x^2 - 2x + 1}\,dx$$

$$= \frac{x^2}{2} + 2x + \int \frac{3x-2}{(x-1)^2}\,dx$$

$$\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$\Rightarrow 3x - 2 = A(x-1) + B$$
$$= Ax + (B - A)$$

$$\Rightarrow A = 3, \quad B - A = -2$$

$$\Rightarrow B = 1$$

$$= \frac{x^2}{2} + 2x + \int \frac{3}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$$

ANSWER:

$$\boxed{\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C}$$

10. Determine if the following improper integrals converge or diverge. Justify your work completely, with correct limit work (no plugging in ∞ as an argument into functions!) when needed.

(a) (10 points) $\int_1^\infty e^{-x}\,dx$

$$= \lim_{t\to\infty} \int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left(-e^{-x}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-e^{-t} - (-e^{-1})\right)$$

$$= \lim_{t\to\infty} \left(-\underbrace{\frac{1}{e^t}}_{\to 0} + \frac{1}{e}\right)$$

$$= \frac{1}{e}$$

ANSWER:

Converges

(b) (10 points) $\int_0^{100} \frac{1}{x^{1/5}}\,dx$

$$= \lim_{t \to 0^+} \int_t^{100} \frac{1}{x^{1/5}}\,dx$$

$$= \lim_{t \to 0^+} \int_t^{100} x^{-1/5}\,dx$$

$$= \lim_{t \to 0^+} \frac{5}{4} x^{4/5} \Big|_t^{100}$$

$$= \lim_{t \to 0^+} \left(\frac{5}{4} 100^{4/5} - \underbrace{\frac{5}{4} t^{4/5}}_{\to 0} \right)$$

$$= \frac{5}{4} 100^{4/5}$$

ANSWER:

Converges

(c) (10 points) $\int_1^\infty \frac{x^2}{x^6+\ln(x)}\,dx$

$$0 \le \frac{x^2}{x^6+\ln(x)} \le \frac{x^2}{x^6} = \frac{1}{x^4}$$

$\int_1^\infty \frac{1}{x^4}\,dx$ converges by p-test $(p=4)$

$\Rightarrow \int_1^\infty \frac{x^2}{x^6+\ln(x)}$ converges by the comparison test

ANSWER:

Converges

11. (10 points) Find the length of the curve given by $y = 2x^{3/2}$ for $0 \leq x \leq 1$.

$$\frac{dy}{dx} = 2 \cdot \frac{3}{2} x^{1/2} = 3\sqrt{x} \rightarrow \left(\frac{dy}{dx}\right)^2 = 9x$$

$$l = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+9x}\, dx$$

$u = 1+9x$
$du = 9\, dx$

$$\longrightarrow = \int_1^{10} \sqrt{u} \cdot \frac{du}{9}$$

$$= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

$$= \frac{2}{27}\left(10^{3/2} - 1^{3/2}\right)$$

$\cancel{4^{3/2} = (4^{\frac{1}{2}})^3}$ $\cancel{= 2^3 = 8}$

$\cancel{= \frac{2}{9}(8-1)}$

$\cancel{= \frac{14}{9}}$

ANSWER:

$\cancel{\frac{14}{9}}$ $\boxed{\frac{2}{27}\left(10^{3/2} - 1\right)}$