MATH 142, SUMMER 2021 (A2)

MIDTERM 2
SOLUTIONS

2. (10 points)

A particle travels with velocity $f(t) = 1/(1+t)$ in m/s while travelling along a line for 6 seconds, starting at $t = 0$. Find the time at which the instantaneous velocity of the particle is equal to its average velocity over its motion.

$$\text{AVERAGE VELOCITY} = \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{6-0}\int_0^6 \frac{dt}{t+1}$$

$$= \frac{1}{6}\int_1^7 \frac{du}{u} \quad [u = t+1]$$

$$= \frac{1}{6}\ln u\Big]_1^7 = (\ln 7)/6$$

$$f(c) = (\ln 7)/6 \Rightarrow \frac{1}{1+c} = \frac{\ln 7}{6}$$

$$\Rightarrow \boxed{c = \frac{6}{\ln 7} - 1}$$

ANS

3. (30 points) Consider the region bounded in the first quadrant by $y = x^2$ and $y = \sqrt{x}$.

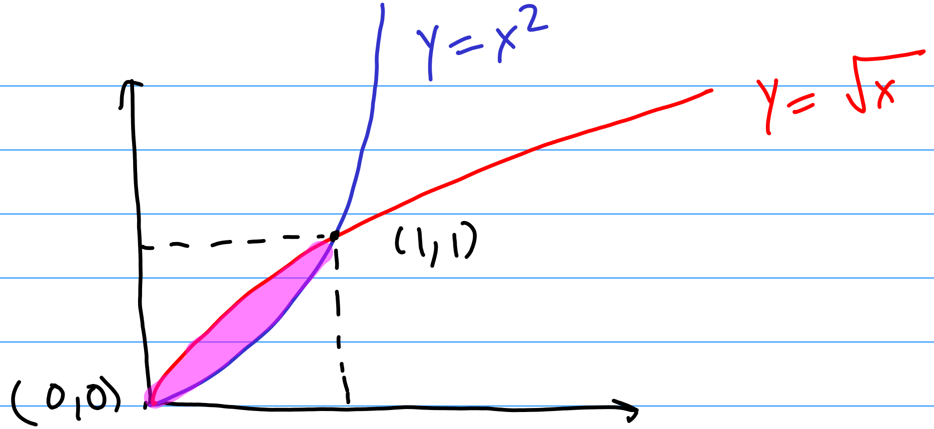
(a) Sketch the curves and shade the region described above.

(b) Write (but do NOT evaluate) an integral that is equal to the area of the region.

(c) Write (but do NOT evaluate) an integral using the shell method for the volume of the solid obtained by revolving the region about the y-axis.

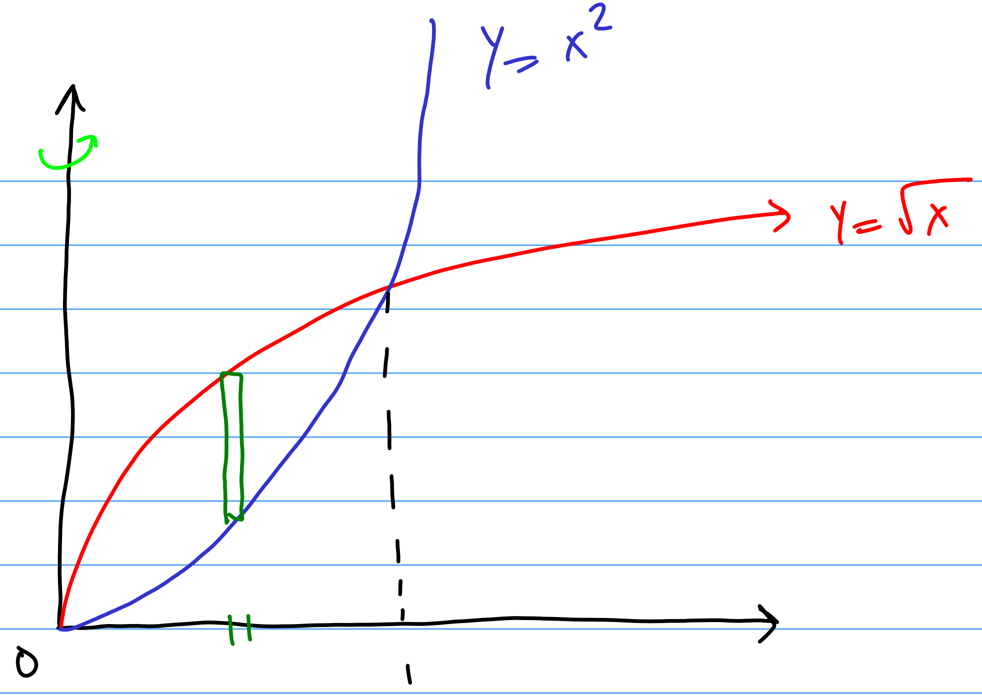
(d) Write (but do NOT evaluate) an integral using the washer method for the volume of the solid obtained by revolving the region about the line $x = 1$.

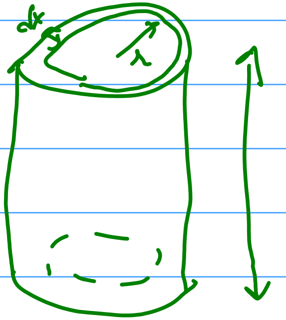
(a)



(b) $\int_0^1 \left(\sqrt{x} - x^2\right) dx$

(c)



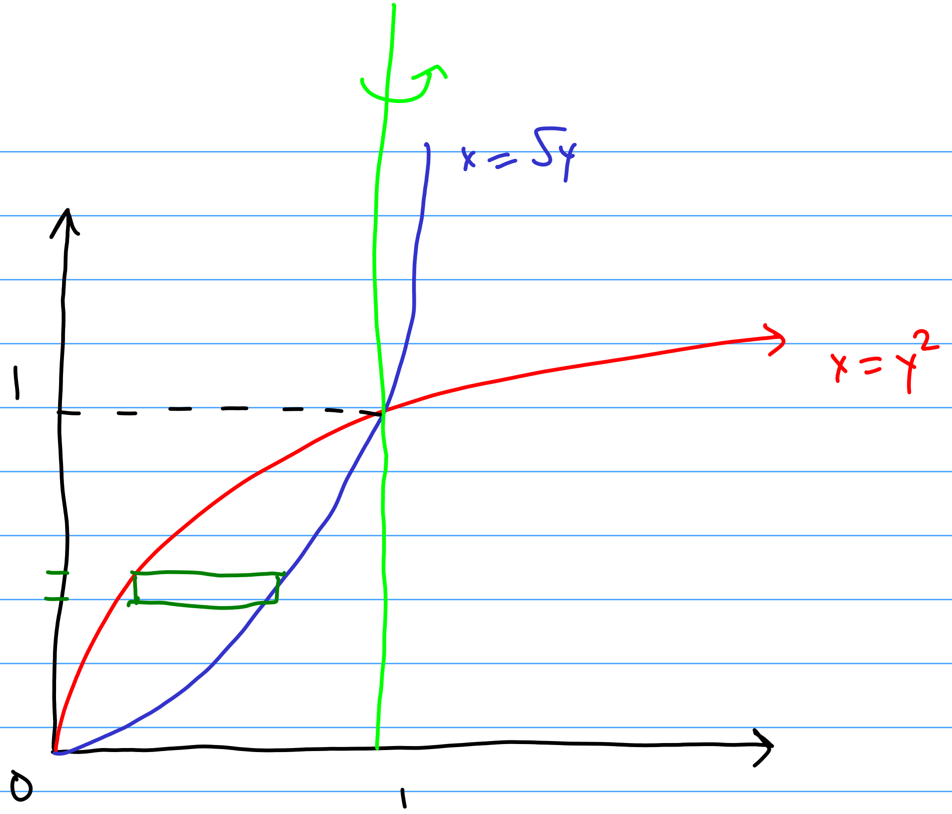


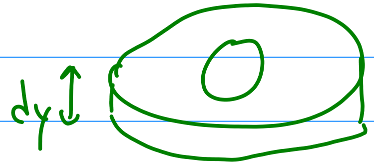
$h = \sqrt{x} - x^2$

$r = x$

$$\text{Vol} = \int_0^1 (2\pi x)(\sqrt{x} - x^2)\, dx$$

(d)





$r_{out} = 1 - y^2$

$r_{in} = 1 - \sqrt{y}$

$$VOL = \int_0^1 \pi \left[(1-y^2)^2 - (1-\sqrt{y})^2 \right] dy$$

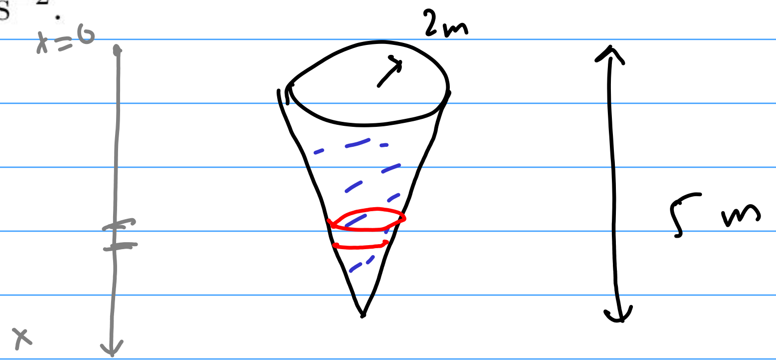
4. (20 points)

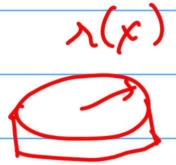
An inverse conical tank of radius $r = 2\,\mathrm{m}$ and height $h = 5\,\mathrm{m}$ is full of water. The water is pumped out of a hole at the top of the tank over time.

(a) Write (but do NOT evaluate) an integral that represents the work done to bring the water level down to a height of $2\,\mathrm{m}$.

(b) Write (but do NOT evaluate) an integral that represents the work done to empty the tank.

Recall that the density of water is $1000\,\mathrm{kg\,m^{-3}}$ and that the acceleration due to gravity is $9.8\,\mathrm{m\,s^{-2}}$.





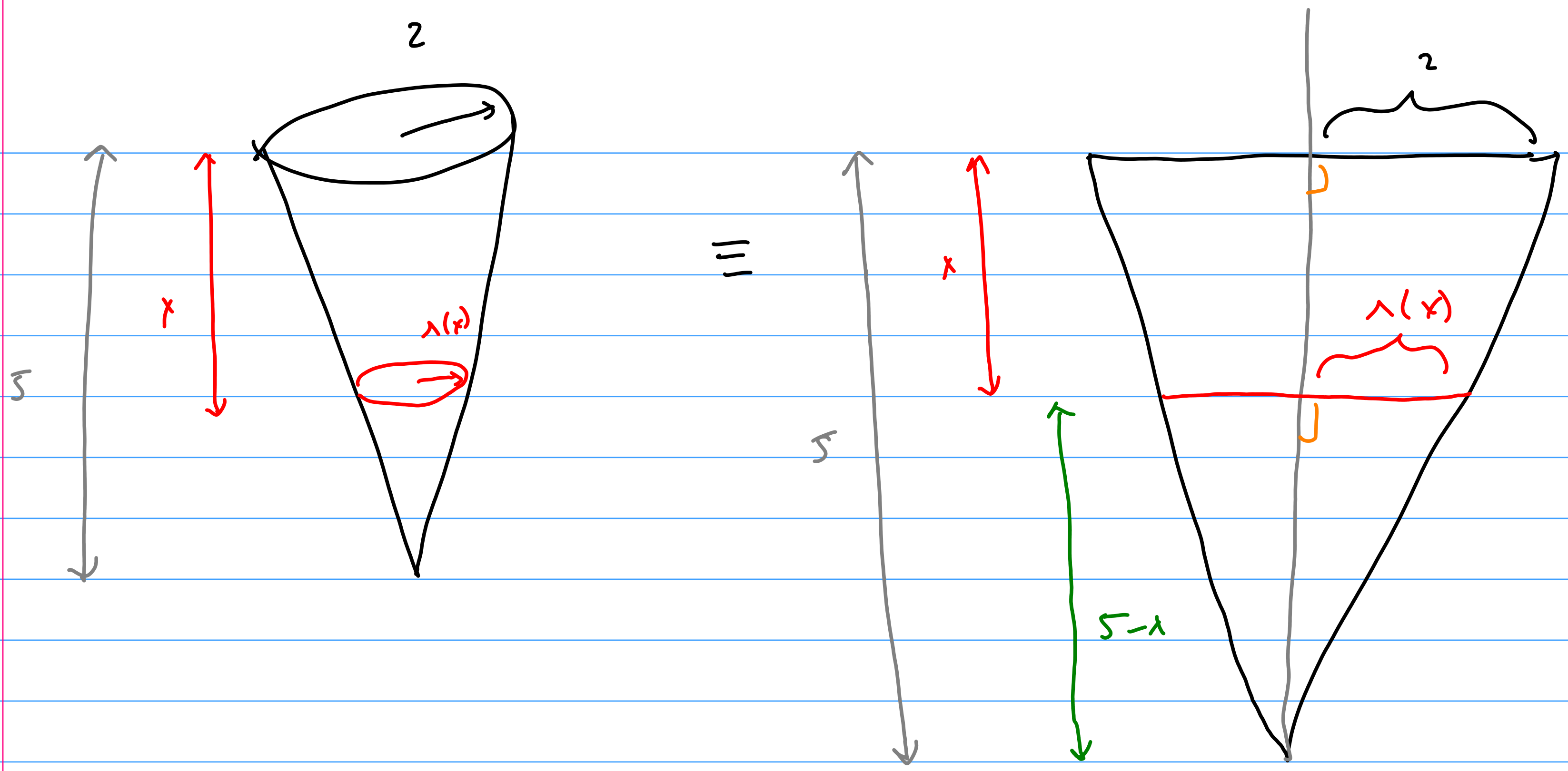
$V = \pi r^2 dx$

(MASS) $m = V\rho = (\pi r^2 \, dx)(1000)$

(FORCE) $F = mg = (\pi r^2 \, dx)(1000)(9.8)$

(WORK) $\Delta W = (F)(\underbrace{\text{DISTANCE}}_{x \text{ BY PICTURE}}) = 9800 \pi r^2 x \, dx$

NEED TO FIND r!



BY SIMILAR Δs : $\frac{r(x)}{2} = \frac{5-x}{5} \Rightarrow r(x) = \frac{10-2x}{5}$

HEIGHT 2 m LEFT $\Rightarrow$ TOP 3 m CLEARED

$$\therefore \text{WORK} = (9800\pi) x \left(\frac{10-2x}{5}\right)^2 dx$$

3

(a) $$\int_0^3 (9800\pi) x \left(\frac{10-2x}{5}\right)^2 dx$$

(b) $$\int_0^5 (9800\pi) x \left(\frac{10-2x}{5}\right)^2 dx$$

5. (30 points)

Compute the following indefinite integrals:

(a)

$$\int \frac{(x+2x^3)}{(x^2+x^4)^3}\,dx$$

(b)

$$\int e^u \cos u\,du$$

(c)

$$\int t^2 e^t\,dt$$

(a) $$\int \frac{x + 2x^3}{(x^2 + x^4)^3} \, dx$$

LET $u = x^2 + x^4 \Rightarrow du = (2x + 4x^2)\, dx$

$$\Rightarrow \frac{du}{2} = (x + 2x^2)\, dx$$

$$\therefore \int \frac{x + 2x^3}{(x^2 + x^4)^3} \, dx = \int \frac{du}{2u^3} = -\frac{1}{4u^2} + C$$

$$= -\frac{1}{4(x^2+x^4)^2} + C$$

(b) $\int e^u \cos u \, du$

$U = \cos u \Rightarrow dU = (-\sin u)\,du$

$dV = e^u \, du \Rightarrow V = e^u$

$\therefore \quad e^u \cos u - \int e^u (-\sin u)\,du$

$= e^u \cos u + \int e^u \sin u \, du$

$U = \sin u \Rightarrow dU = (\cos u)\,du$

$dV = e^u \, du \Rightarrow V = e^u$

$$I = \int e^{u} \cos u \, du = e^{u} \cos u + \left(e^{u} \sin u - \int e^{u} \cos u \, du \right)$$

$$\Rightarrow \quad I = e^{u} (\cos u + \sin u) - I$$

$$\Rightarrow \quad I = \frac{1}{2} e^{u} (\cos u + \sin u) + C$$

(c) $\int t^2 e^t \, dt$

$u = t^2 \Rightarrow du = 2t \, dt$

$dv = e^t \, dt \Rightarrow v = e^t$

$\therefore \quad t^2 e^t - \int 2t \, e^t \, dt$

$u = 2t \Rightarrow du = 2$

$dv = e^t \, dt \Rightarrow v = e^t$

$$\therefore \quad t^2 e^t - \left(2t e^t - \int 2e^t \, dt \right)$$

$$= t^2 e^t - 2te^t + 2e^t + C$$

6. (30 points)

Compute the following definite integrals:

(a)

$$\int_1^2 \frac{(\ln v)^2}{v}\, dv$$

(b)

$$\int_0^{2\pi} x \cos x \, dx$$

(c)

$$\int_0^1 \arctan x \, dx$$

(a) $$\int_1^2 \frac{(\ln v)^2}{v}\, dv$$

$$u = \ln v \Rightarrow du = \frac{dv}{v}$$

$$u(1) = \ln 1 = 0$$
$$u(2) = \ln 2$$

$$\therefore \int_0^{\ln 2} u^2\, du = \frac{u^3}{3}\Big]_0^{\ln 2} = (\ln 2)^3$$

(b) $\displaystyle\int_0^{2\pi} x \cos x \, dx$

$$\left(\begin{array}{l} \sin 2\pi = \sin 0 = 0 \\ \cos 2\pi = \cos 0 = 1 \end{array} \right)$$

$u = x \Rightarrow du = dx$

$dv = \cos x \, dx \Rightarrow v = \sin x$

$$\therefore \quad x \sin x \Big]_0^{2\pi} - \int_0^{2\pi} \sin x \, dx$$

$$= 2\pi \sin 2\pi - 0 \sin 0 - \Big[-\cos x \Big]_0^{2\pi} = 0 - [-1 + 1] = 0$$

(c) $\displaystyle\int_0^1 \operatorname{Arctan} x \, dx$

$u = \operatorname{Arctan} x \Rightarrow du = \dfrac{dx}{1+x^2}$

$dv = dx \Rightarrow v = x$

$\left(\operatorname{Arctan} 1 = \dfrac{\pi}{4} \right)$

$\therefore \; x \operatorname{Arctan} x \Big]_0^1 - \displaystyle\int_0^1 \frac{x \, dx}{1+x^2}$

$= \operatorname{Arctan} 1 - \displaystyle\int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \int_0^1 \frac{x \, dx}{1+x^2}$

$u = 1 + x^2 \Rightarrow du = 2x\,dx$

$\Rightarrow \frac{du}{2} = x\,dx$

$u(0) = 1 + 0^2 = 1$

$u(1) = 1 + 1^2 = 2$

$$\therefore \quad \frac{\pi}{4} - \int_1^2 \frac{du}{2u} = \frac{\pi}{4} - \frac{\ln u}{2}\Big]_1^2$$

$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$