Math 142: Calculus II

Midterm 2

November 16, 2017

NAME (please print legibly): Solutions

Your University ID Number: ____________________

Indicate your instructor with a check in the appropriate box:

Crossen	MW 9-10:15	
Zhong	MW 3:25-4:40	

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 8 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________________

QUESTION	VALUE	SCORE
1	15	
2	15	
3	10	
4	10	
5	10	
6	15	
TOTAL	75	

1. (15 points) Evaluate the following indefinite integrals.

(a) $\int x^3 \ln x \, dx.$ $= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \left(\frac{1}{x}\right) dx$

$u = \ln x$
$dv = x^3 dx$
$du = \frac{1}{x} dx$
$v = \frac{1}{4}x^4$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$

$= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$

(b) $\int \sec^4 x \, dx.$ $= \int \sec^2 x \cdot \sec^2 x \, dx$

$= \int (1 + \tan^2 x) \sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int (1 + u^2) du$

$= u + \frac{1}{3}u^3 + C$

$= \boxed{\tan x + \frac{1}{3}\tan^3 x + C}$

2. (15 points) Evaluate the following definite integrals.

(a) $\int_0^1 \frac{x^2}{1+x^6}\,dx.$ $= \int_0^1 \frac{x^2 dx}{1+(x^3)^2} = \frac{1}{3}\int_0^1 \frac{du}{1+u^2}$

$u = x^3$

$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

$u(0) = 0$

$u(1) = 1$

$= \frac{1}{3} \arctan u \Big|_0^1$

$= \frac{1}{3}\arctan(1) - \frac{1}{3}\arctan(0)$

$= \frac{1}{3}(\pi/4) - 1/3(0)$

$= \boxed{\pi/12}$

(b) $\int_0^{\pi/6} \sin^3 x\,dx.$ $= \int_0^{\pi/6} \sin^2 x \cdot \sin x\,dx$

$= \int_0^{\pi/6} (1-\cos^2 x)\sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$u(0) = \cos(0) = 1$

$u(\pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}$

$= -\int_1^{\sqrt{3}/2} (1-u^2)\,du$

$= -\left[u - \frac{1}{3}u^3\right]_1^{\sqrt{3}/2}$

$= -\left[\frac{\sqrt{3}}{2} - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3\right] + \left[1 - \frac{1}{3}(1)^3\right]$

$= -\frac{\sqrt{3}}{2} + \frac{1}{\cancel{3}} \cdot \frac{3^{3/2 \; 1/2}}{8} + 1 - 1/3$

$-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} + 2/3$

$\frac{-4\sqrt{3}+\sqrt{3}}{8} + 2/3 = \boxed{-\frac{3\sqrt{3}}{8} + 2/3}$

3. (10 points) Consider the function $f(x) = 3x^2 - 12x - 10$.

(a) For $b > 0$, compute the average value of $f(x)$ on the interval $0 \leq x \leq b$. **Note:** Your answer should be a function of b.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{b-0}\int_0^b (3x^2-12x-10)\,dx$$

$$= \frac{1}{b}\left[x^3-6x^2-10x\right]_0^b$$

$$= \frac{1}{b}\left[b^3-6b^2-10b\right]$$

$$= \boxed{b^2-6b-10}$$

(b) Find all numbers $b > 0$ such that the average value of $f(x)$ on $[0, b]$ is equal to 6.

$$f_{ave} = 6$$

$$b^2-6b-10 = 6$$

$$b^2-6b-16=0$$

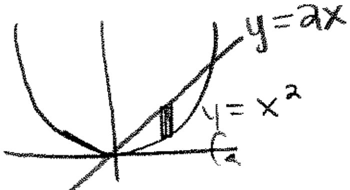
$$(b-8)(b+2)=0$$

$$b=-2, 8$$

$$b>0 \Rightarrow \boxed{b=8}$$

4. (10 points) The following problems concern the solid of revolution generated by rotating about a given axis the region R, which is enclosed by the curve $y = x^2$ and the curve $y = 2x$. You may use either the method of disks/washers or the method of cylindrical shells, but you must clearly indicate which one you are using in each problem.

(a) If R is rotated about the x-axis, set up but do not evaluate an integral for computing the volume of the resulting solid.



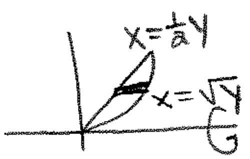
$2x = x^2 \Rightarrow x = 0, 2$

$x^2 - 2x = 0 \quad y = 0, 4$

$x(x-2) = 0$

Washers: $V = \pi \int_a^b (r_{outer}^2 - r_{inner}^2)\,dx$

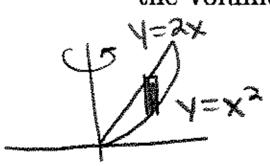
$= \pi \int_0^2 \left[(2x)^2 - (x^2)^2\right] dx$



Shells: $V = 2\pi \int_c^d r h\,dy$

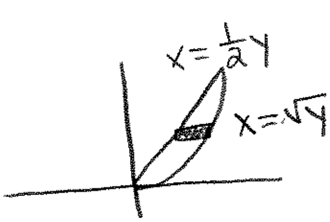
$= 2\pi \int_0^4 y(\sqrt{y} - \frac{1}{2}y)\,dy$

(b) If R is rotated about the y-axis, set up but do not evaluate an integral for computing the volume of the resulting solid.



Shells: $V = 2\pi \int_a^b r h\,dx$

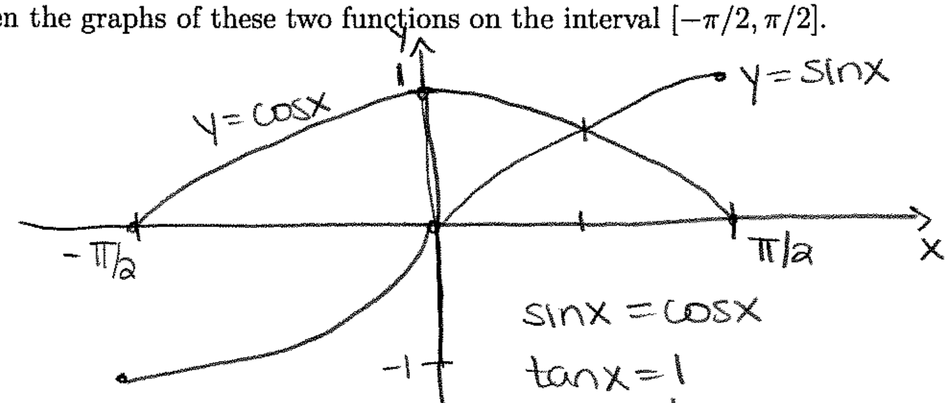
$= 2\pi \int_0^2 x\,(2x - x^2)\,dx$



Washers: $V = \pi \int_c^d (r_{outer}^2 - r_{inner}^2)\,dy$

$= \pi \int_0^4 \left((\sqrt{y})^2 - (\frac{1}{2}y)^2\right) dy$

5. (10 points) Consider the functions $f(x) = \sin x$ and $g(x) = \cos x$. Compute the area between the graphs of these two functions on the interval $[-\pi/2, \pi/2]$.



$\sin x = \cos x$

$\tan x = 1$

$x = \pi/4$

$$\int_{-\pi/2}^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

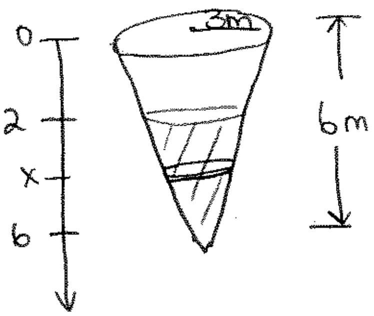
$$= \Big[\sin x + \cos x\Big]_{-\pi/2}^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left[\sin(\pi/4) + \cos(\pi/4)\right] - \left[\sin(-\pi/2) + \cos(-\pi/2)\right] + \left[-\cos(\pi/2) - \sin(\pi/2)\right] - \left[-\cos(\pi/4) - \sin(\pi/4)\right]$$

$$= \left[1/\sqrt{2} + 1/\sqrt{2}\right] - \left[(-1) + 0\right] + \left[0 - (1)\right] - \left[-1/\sqrt{2} - 1/\sqrt{2}\right]$$

$$= 4/\sqrt{2} = \boxed{2\sqrt{2}}$$

6. (15 points) A tank has the shape of an inverted circular cone with height 6m and base radius 3m. It is filled with water to a height of 4m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)



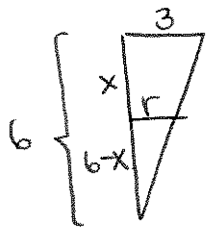
$F_{slice} = (1000\ kg/m^3)(9.8\ m/s^2)(V_{slice}\ m^3)$

$V_{slice} = \pi r_{slice}^2 \Delta x\ m^3$

$r_{slice} = ? = \frac{1}{2}(6-x)$

$F_{slice} = 9800\pi \left(\frac{1}{2}(6-x)\right)^2 \Delta x\ N$

$d_{slice} = x\ m$



$\frac{3}{r} = \frac{6}{6-x}$

$6r = 3(6-x)$

$r = \frac{1}{2}(6-x)$

$$W = \int_2^6 \left(9800\pi \left(\frac{1}{2}(6-x)\right)^2 x\, dx\right.$$

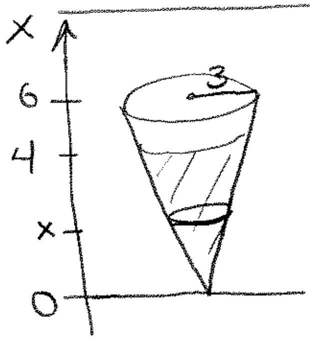
$$= \frac{9800\pi}{4} \int_2^6 (36 - 12x + x^2)\, x\, dx$$

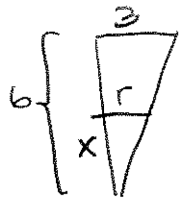
$$= \frac{9800\pi}{4} \int_2^6 (36x - 12x^2 + x^3)\, dx$$

$$= \frac{9800\pi}{4} \left[18x^2 - 4x^3 + \frac{1}{4}x^4\right]_2^6$$

$$= \boxed{\frac{9800\pi}{4}\left[\left(18(6)^2 - 4(6)^3 + \frac{1}{4}(6)^4\right) - \left(18(2)^2 - 4(2)^3 + \frac{1}{4}(2)^4\right)\right] J}$$

Alternate #6





$$\frac{3}{r} = \frac{6}{x}$$
$$6r = 3x$$
$$r = \frac{1}{2}x$$

$$F_{slice} = (1000\ kg/m^3)(9.8\ m/s^2)(V_{slice}\ m^3)$$

$$V_{slice} = \pi r_{slice}^2\ \Delta x\ \ m^3$$

$$r_{slice} = \frac{1}{2}x$$

$$F_{slice} = 9800\pi\left(\frac{1}{2}x\right)^2 \Delta x\ \ N$$

$$d_{slice} = 6 - x$$

$$W = \int_0^4 9800\pi\left(\frac{1}{2}x\right)^2(6-x)\,dx$$

$$= \frac{9800\pi}{4}\int_0^4 (6x^2 - x^3)\,dx$$

$$= \frac{9800\pi}{4}\left[2x^3 - \frac{1}{4}x^4\right]_0^4$$

$$= \frac{9800\pi}{4}\left[2(4)^3 - \frac{1}{4}(4)^4\right]$$

$$= \boxed{9800\pi(4)^2\ J}$$